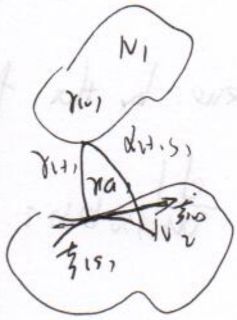
HW6. Rie. Geo. 1

1. By the assumption. We need to show. $\forall\, \tilde{T} \in T_{\gamma(a)}N_2$

$$\langle \dot\gamma(a), \tilde{T} \rangle_g = 0. \quad g \text{ is the metric on } M.$$



choose a geodesic $\xi(s)$ s.t. $\xi(0) = \gamma(a)$. $\dot\xi(0) = \tilde{T}$.

Since γ is a shortest geodesic. we know it is a

critical point of the length functional.

Hence. we construct a variation near γ.

let $\alpha(t,s)$ be the curve that connecting $\gamma(0)$ and $\xi(s)$.

To be convient, denote $T = d\alpha(\frac{\partial}{\partial t})|_{s=0}$. $V = d\alpha(\frac{\partial}{\partial s})|_{s=0}$

We then have. $E(s) = \int_0^a \langle d\alpha(\frac{\partial}{\partial t}), d\alpha(\frac{\partial}{\partial t}) \rangle dt$

Thus. $\frac{d}{ds}\Big|_{s=0} E(s) = \int_0^a \frac{d}{ds} \langle d\alpha(\frac{\partial}{\partial t}), d\alpha(\frac{\partial}{\partial t}) \rangle|_{s=0}\, dt$

$$= \int_0^a 2\langle \tilde\nabla_V T, T \rangle\, dt$$

$$= \int_0^a 2\langle \tilde\nabla_T V, T \rangle dt$$

$$= 2\int_0^a \left(T\langle V, T \rangle - \langle V, \tilde\nabla_T T \rangle \right) dt$$

$$= 2\langle V, T \rangle \Big|_0^a$$

Here since γ is a geodesic. $\tilde\nabla_T T = 0$.

note. that $V(0) = 0$. since ~~p~~ ~~~~ for $\gamma(0)$ is a fixed point

$\Rightarrow \frac{d}{ds}\Big|_{s=0} E(s) = 2\langle V(a), T(a) \rangle$.

but $T(a) = \dot\gamma(a)$. $V(a) = \dot\xi(0) = \tilde{T}$.

it follows that $0 = \frac{d}{ds}\Big|_{s=0} E(s) = 2\langle \tilde{T}, \dot\gamma(a) \rangle$ i.e. $\dot\gamma(a) \perp T_{\gamma(a)}N_2$

Similarly. ~~~~ $\dot\gamma(0) \perp T_{\gamma(0)}N_1$

2. The energy functional is change as.

$$\tilde{E}(s) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \langle d f_i(\tfrac{\partial}{\partial t}), d f_i(\tfrac{\partial}{\partial t}) \rangle_g \, dt$$

Hence. by the first variational formula for smooth curves. we have.

$$\frac{d}{ds}\Big|_{s=0} \tilde{E}(s) = \sum_{i=0}^{k} \left(\langle V, T \rangle \Big|_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} \langle \tilde{\nabla}_T \langle V, \tilde{\gamma}_1 \rangle \, dt \right).$$

$$= \sum_{i=0}^{k} \langle V, T \rangle \Big|_{t_i}^{t_{i+1}} - \int_0^a \langle V, \tilde{\gamma}_1 \rangle \, dt$$

(ii). ~~This qu~~ I have no better method.

As usual. choose $\xi(s)$. s.t. $\xi(0) = c(0)$. $\dot{\xi}(0) = \cancel{V(0)} \frac{V(0)}{|V(0)|}$. $\nabla_{\dot{\gamma}(0)} V = W$.

then parallel transport ~~V~~ $\dot{c}(0)$ and W along $\xi(s)$. we have two parallel vector field. ~~V(s) W(s)~~. $T(s)$. $W(s)$.

~~Consider the variation $\alpha(t,s) = \exp_{\xi(s)} t(V(s) + sW(s))$.~~

~~Obviously. $\alpha(t,0) = \exp_{\xi(0)} +$~~

consider the variation $\alpha(t,s) = \exp_{\xi(s)} t(T(s) + sW(s))$.

Obviously. $\alpha(t,0) = \exp_{\xi(0)} t\,T(0) = c(t)$.

We want to show the variation field of $\alpha(t,s)$ ~~is V~~ V_1 is exactly V.

~~Here~~ Indeed. we have.

$$V_1(0) = \frac{\partial}{\partial s} \alpha(t,s)\Big|_{s=0,\, t=0} = \frac{d}{ds}\Big|_{s=0} \xi(s) = \dot{\xi}(0) = V(0).$$

and

$$\dot{V}_1(0) = \frac{\partial}{\partial t}\frac{\partial}{\partial s} \alpha(t,s)\Big|_{s=0,\, t=0} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} \alpha(t,s)\Big|_{s=0,\, t=0}.$$

$$= \frac{\partial}{\partial s} (T(s) + sW(s))\Big|_{s=0}$$

$$= \cancel{\ } + \tilde{\nabla}_{\dot{\xi}(0)} T(s)\Big|_{s=0} + W(0) + s\, \tilde{\nabla}_{\dot{\xi}(0)} W(s)\Big|_{s=0}$$

$$= W(0)$$

Since $T(s)$ and $W(s)$ are both parallel.

This shows us that $V(t)$ has the same initial direction. 2

as $\bar{E}.V$ hence $V_1 = V$.

Moreover, if $V(0) = V(a) = 0$.

We can choose ~~$\alpha(t,s) = \exp_{c(0)}(\ldots(s) + sW(s))$~~.

(ii). Intuitively, at any point $c(t)$, we can construct a geodesic with initial direction $V(t)$.

Then set $\alpha(t,s) = \exp_{c(t)} sV(t)$.

we have $\alpha(t,0) = c(t)$. $\frac{\partial}{\partial s}\alpha(t,s)|_{s=0} = V(t)$.

If $V(0) = V(a) = 0$ then we have $\alpha(0,s) = c(0)$, $\alpha(a,s) = c(a)$.

(iii).

Recall the first variation formula. for a piecewise smooth curve.

$$\frac{d}{ds}\Big|_{s=0} E(s) = \frac{d}{ds}\Big|_{s=0} E(s) = -\int_0^a \langle V, \tilde{\nabla}_1 T\rangle dt + \langle V, T\rangle\Big|_0^a + \sum_{i=1}^{k-1}\langle V, T\rangle\Big|_{t_i}^{t_{i+1}}$$

Since f is proper, we have $V(0) = V(a) = 0$.

hence $\frac{d}{ds}\Big|_{s=0} E(s) = \sum_{i=1}^{k-1}\langle V, T\rangle\Big|_{t_i}^{t_{i+1}} - \int_0^a \langle V, \tilde{\nabla}_1 T\rangle dt$

Here V is a piecewise smooth vector field. and.

$$V(t_{i+1}) = \lim_{t\to t_{i+1}} V(t), \quad V(t_i) = \lim_{t\to t_i} V(t),$$

We now want to choose some special variation fields to obtain ~~such~~ the result.

"$\Leftarrow$". Suppose first $E(0) = 0$. let by (ii). any piecewise smooth vector field along c, there always exists a variation ~~field~~ whose ~~vector~~ variation field is the given variation field.

Hence, we chose $V(t) = \cancel{\lambda(t)\dot\gamma(t)}\ \lambda(t)\tilde{\nabla}_{\dot\gamma}T$.

where $\cancel{\lambda(t)>0}$ $\lambda(t_i) = 0$ and $\lambda(t) > 0$, $t \neq t_i$.

then we have $V(t_i) = 0$ and

$$0 = E'(0) = -\int_0^a \lambda(t)|\tilde{\nabla}_{\dot\gamma}T|^2\,dt$$

this implies $\tilde{\nabla}_{\dot\gamma}T = 0$ for $t \neq t_i$

i.e. γ is a broken geodesic.

Next we ~~ca~~ further show ~~V(t_i)~~ $\dot c(t_i^+) - \dot c(t_i^-) = 0$.

Set $V = \cancel{\lambda(t)\dot\gamma(t)}.\ \cancel{\lambda(t)}\ \lambda'(t)\gamma(t)$

write $\frac{d}{ds}\Big|_{s=0} E(s) = \langle \cancel{V(t)} T(t), \sum_{i=0}^{k} \langle V, T\rangle \Big|_{t_i}^{t_{i+1}}$

$$= \cancel{\gamma}\langle V(t_{k+1}), T(t_{k+1})\rangle - \langle V(t_k), T(t_k)\rangle + \langle V(t_{k-1}), T(t_{k-1})\rangle - \cdots$$

$$- \langle V(t_{k-1}), T(t_{k-1})\rangle + \cdots + \langle V(t_1), T(t_1)\rangle - \langle V(t_0), T(t_0)\rangle.$$

$$= \langle \cancel{V, T}\rangle\Big|_a^b$$

$$= \langle V, T\rangle\Big|_0^a + \sum_{i=1}^{k-1} \langle V(t_i), T(t_i^+) - T(t_i^-)\rangle$$

We set $\lambda'(t_i) = T(t_i^+) - T(t_i^-)$

where $T(t_i^+) = \lim_{t\to t_i^+} T(t) = \dot c(t_i^+)$ & $T(t_i^-) = \lim_{t\to t_i^-} T(t) = \dot c(t_i^-)$

we have. $\quad 0 = E'(0) = \sum_{i=1}^{k-1} |T(t_i^+) - T(t_i^-)|^2$

$\Rightarrow T(t_i^+) - T(t_i^-) = \dot c(t_i^+) - \dot c(t_i^-) = 0$

hence γ is indeed a smooth geodesic.

$\Rightarrow$ if γ is a geodesic. then. $\tilde{\nabla}_T T = 0$.

and γ is necessarily smooth. hence.

$\dot{c}(t_i^+) - \dot{c}(t_i^-) = 0$. and.

$$\frac{d}{ds}\Big|_{s=0} \bar{E}(s) = \sum_{i=1}^{k-1} \langle \dot{c}(t_i^+) - \dot{c}(t_i^-), V(t_i) \rangle = 0.$$

HW7.

1. We calculate in local coordinates.

suppose $s(0,0) = p \in M$. and $x^1, \cdots, x^n$ is a local coordinate

around p. then locally.

$$\frac{\partial s}{\partial x} = \frac{\partial x^i}{\partial x} \frac{\partial}{\partial x^i}. \quad \frac{\partial s}{\partial y} = \frac{\partial x^i}{\partial y} \frac{\partial}{\partial x^i}.$$

Set V_1