§5. The Second Variation: Revisited. [JJ, §4.1] [WSY, Chap. 6]

Recall from §1 that the curvature tensor is closedly related to the 2^{nd} variation of the energy functional (and the length functional) of a normal geodesic. In this section, we will discuss some geometric and topological implications ~~when~~ assuming curvature restrictions via applying SVF.

Let γ be a normal geodesic, i.e. $|\dot\gamma|=1$. Consider a variation

$$F : [a,b]\times(-\varepsilon,\varepsilon) \to M$$
$$(t, v) \longmapsto F(t,v)$$

(i.e. F is smooth and $F(t,0)=\gamma(t)$).

The variational field $V(t) = \frac{\partial F}{\partial v}(t,0)$ is a vector field along γ.

<u>Definition 6</u> (geodesic variation). The variation F is called <u>a geodesic variation</u> if each curve $\gamma_v(t) := F(t,v)$ is a geodesic.

Next, we recall briefly the second variation formula from §1. For the one-parameter family of curves $\{\gamma_v\}_{v\in(-\varepsilon,\varepsilon)}$, we have $E(v) := E(\gamma_v)$ be a function on $(-\varepsilon,-\varepsilon)$. Since $\gamma_0=\gamma$ is a geodesic, we have $E'(0)=0$.

$$E''(0) \quad \frac{\partial^2}{\partial v^2}E(v) = \int_a^b \left(\left\langle \tilde\nabla_{\frac{\partial}{\partial v}} \tilde\nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \tilde\nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \tilde\nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v} \right\rangle \right) dt$$

(Excercise. Recall from §1).

$$= \int_a^b \left\langle R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle + \left\langle \tilde\nabla_{\frac{\partial}{\partial t}} \tilde\nabla_{\frac{\partial}{\partial v}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle + \left\langle \tilde\nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \tilde\nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}\right\rangle dt$$

$$= -\int_a^b \langle R(\tfrac{\partial F}{\partial v}, \tfrac{\partial F}{\partial t})\tfrac{\partial F}{\partial t}, \tfrac{\partial F}{\partial v}\rangle \, dt$$

$$+ \int_a^b \frac{\partial}{\partial t}\langle \tilde{\nabla}_{\frac{\partial}{\partial v}}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle - \langle \tilde{\nabla}_{\frac{\partial}{\partial v}}\frac{\partial F}{\partial v}, \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t}\rangle \, dt$$

$$+ \int_a^b \langle \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial v}, \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial v}\rangle \, dt.$$

<u>Proposition 14</u>: Let F be a <u>geodesic</u> variation of a curve $\gamma:[a,b]\to M$.

Then
$$\frac{\partial^2}{\partial v^2}E(v) = \int_a^b \left(\langle \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial v}, \tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial v}\rangle - \langle R(\tfrac{\partial F}{\partial v}, \tfrac{\partial F}{\partial t})\tfrac{\partial F}{\partial t}, \tfrac{\partial F}{\partial v}\rangle\right) dt.$$

<u>Proof</u>: Use the fact $\tilde{\nabla}_{\frac{\partial}{\partial v}}\frac{\partial F}{\partial v} = 0$ since F is a geodesic variation. $\square$

In particular, for a geodesic variation of a normal geodesic $\gamma:[a,b]\to M$, we have

$$\boxed{\begin{aligned}&\frac{\partial^2}{\partial v^2}\Big|_{v=0} E(v) := E''(0)\\ &= \int_a^b \left(\langle \nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle\right) dt.\end{aligned}}$$

Observe that when M has nonpositive sectional curvature, we have

$$-\langle R(\tfrac{\partial F}{\partial v}, \tfrac{\partial F}{\partial t})\tfrac{\partial F}{\partial t}, \tfrac{\partial F}{\partial v}\rangle = \underbrace{-K(\tfrac{\partial F}{\partial v}, \tfrac{\partial F}{\partial t})\, G(\tfrac{\partial F}{\partial v}, \tfrac{\partial F}{\partial t}, \tfrac{\partial F}{\partial v}, \tfrac{\partial F}{\partial t})}_{\geq 0}$$

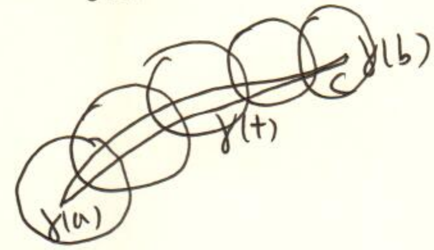
Hence $\frac{\partial^2}{\partial v^2}E(v) \geq 0$ for $v \in (-\varepsilon, \varepsilon)$. This tells immediately:

<u>Corollary 1</u>: On a Riemannian manifold with nonpositive sectional curvature, geodesics with fixed endpoints are always <u>locally minimizing</u>.

<u>Remark</u>: Here a geodesic γ is "locally minimizing" means that for this $\gamma:[a,b]\to M$ there exists some $\delta>0$, such that for any smooth curve $c:[a,b]\to M$ with $c(a)=\gamma(a)$, $c(b)=\gamma(b)$, and $d(\gamma(t), c(t)) \leq \delta$ $\forall t \in [a,b]\to M$

we have $\quad E(c) \geq E(\gamma)$.

<u>Proof</u>. For each $t \in [a,b]$, let δ_t be the parameter of the totally ~~maximal radius such~~ normal neighborhood W_t of $\gamma(t)$. (That is, $\forall q \in W_t$, $\exp_q$ is diffeomorphism on $B(0,\delta_t) \subset T_qM$.)
~~that $\exp_{\gamma(t)}$ is diffeomorphic on $B(0,\delta_t) \subset T_{\gamma(t)}M$.~~



Since $\gamma([a,b])$ is compact, we can find a finite subcover of the cover $\{\exp_{\gamma(t)} B(0,\delta_t)\}_{t\in[a,b]}$. Hence we can find a positive number $\delta > 0$ for $\gamma : [a,b] \to M$ such that

$\delta_t \geq \delta \quad, \forall\, t \in [a,b]$.

Let c be another curve $c : [a,b] \to M$ s.t. $d(\gamma(t), c(t)) \leq \delta$, $\forall\, t \in [a,b]$.

Construct the variation as

$$F(t,s) = \exp_{\gamma(t)} s \exp^{-1}_{\gamma(t)}(c(t)) \quad , \quad t \in [a,b], \; s \in [-1,1]$$

Notice that $F(t,0) = \gamma(t)$, $F(t,1) = c(t)$.

F is a geodesic variation. (and proper).

$\boxed{F \text{ is proper and } \gamma \text{ is a geodesic}} \Rightarrow \boxed{E'(0) = 0}$

$\boxed{F \text{ is a geodesic variation}} \underset{\text{Prop 13}}{\Rightarrow} \boxed{E''(s) \geq 0 \;, s \in [-1,1]}$

Recall Taylor's expansion of an one-variable smooth fct, we have

$$E(1) = E(0) + \underset{0}{\underset{\shortparallel}{E'(0)}} + \underbrace{\int_0^1 (1-t)\, E''(t)\, dt}_{\geq 0}$$

$$\geq E(0).$$

That is $E(c) \geq E(\gamma)$. $\square$

<u>Remark</u>: ① Note that the "locally minimizing energy" also implies "locally minimizing length". From the proof above, for any curve $c : [a,b] \to M$ close to the normal geodesic $\gamma(t)$, we can reparametrize

$c:[a,b]\to M$ s.t. $|\dot{c}(t)| = \frac{l(c)}{b-a}$. Then

$$2(b-a)E(\gamma) = L(\gamma)^2, \quad 2(b-a)E(c) = L(c)^2.$$

$$E(\gamma)\le E(c) \Rightarrow L(\gamma)\le L(c).$$

Of course, one can also argume by using the SVF of the length functional of ~~a~~ curves directly.

<u>Exercise</u>. Let $\gamma:[a,b]\to M$ be a smooth curve, and

$$F: [a,b]\times(-\varepsilon,\varepsilon)\times(-\delta,\delta)\longrightarrow M$$
$$(t, v, w)\longmapsto F(t,v,w)$$

be a 2-parameter variation of γ. Denote by

$$V(t) = \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

the two corresponding variational field.

(i) Show that

$$\frac{\partial^2}{\partial w\partial v}L(v,w) = \int_a^b \frac{1}{\left\|\frac{\partial F}{\partial t}\right\|}\Big\{\langle \tilde\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial v}, \tilde\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial w}\rangle - \langle R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial t}, \frac{\partial F}{\partial v}\rangle$$
$$+ \langle \tilde\nabla_{\frac{\partial}{\partial t}}\tilde\nabla_{\frac{\partial}{\partial w}}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle$$
$$- \frac{1}{\left\|\frac{\partial F}{\partial t}\right\|^2}\left(\langle \tilde\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle\langle \tilde\nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t}\rangle\right\}dt$$

where $\left\|\frac{\partial F}{\partial t}\right\| := \langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\rangle^{1/2}$.

(ii) Let γ be a normal geodesic, i.e. $\|\dot\gamma\|=1$. Show that

$$\frac{\partial^2}{\partial w\partial v}\Big|_{w=v=0} L(v,w) = \int_a^b\Big(\langle\nabla_T V, \nabla_T W\rangle - \langle R(W,T)T,V\rangle - T\langle V,T\rangle T\langle W,T\rangle\Big)dt$$
$$+ \langle \nabla_W V, T\rangle\Big|_a^b$$

where $T(t) := \dot\gamma(t)$ is the velocity field along γ.

(iii). ~~Let~~ Consider the orthogonal component $\tilde V, \tilde W$ of V, W with respect to T, that is

$$V^\perp := V - \langle V,T\rangle T$$
$$W^\perp := W - \langle W,T\rangle T.$$

Show that

$$\frac{\partial^2}{\partial w \partial v}\Big|_{(0,0)} L(v,w) = \int_a^b \left(\langle \nabla_T V^\perp, \nabla_T W^\perp \rangle - \langle R(W^\perp, T)T, V^\perp \rangle \right) dt$$
$$+ \langle \nabla_W V, T \rangle \Big|_a^b . \qquad \square$$

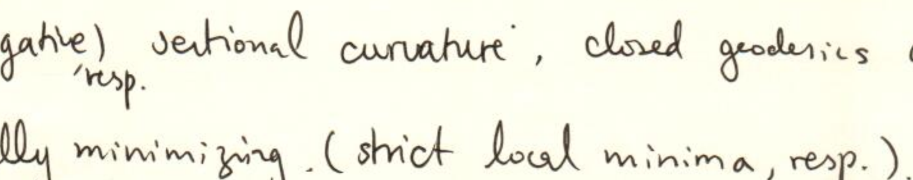
Remark②. Observe in the above proof, the "properness" of the variation F is only used to conclude $E'(0) = 0$. When we consider variation of closed geodesics, i.e. a geodesic

$$\gamma : S^1 \to M$$

where S^1 is the unit circle parametrized by $[0, 2\pi)$.
(in fact. $\gamma : [0, 2\pi] \to M$, $\gamma(0) = \gamma(2\pi)$, $\dot\gamma(0) = \dot\gamma(2\pi)$),
~~We can drop~~ the argument in the proof still works.

Corollary 2: On a Riemannian manifold with nonpositive (negative) resp. sectional curvature, closed geodesics are locally minimizing. (strict local minima, resp.).

Notice that on a manifold with vanishing curvature, closed geodesics are still (locally) minimizing, but not necessarily strictly so any more. On a mfld with positive curvature, closed geodesics in general do not minimize anymore (✱). The following picture is very inspiring.



$K > 0$ $\quad$ S^2 $\qquad$ $K = 0$ $\quad$ $S^1 \times \mathbb{R}$ $\qquad$ $K < 0$

We will derive a global consequence of the fact (✱).

Lecture 14, 2017.04.11

We give a general remark about how ~~what~~ (SVF) ~~tells~~ implies ~~about the geometry / topology~~. minimizing property of geodesics.

Let $\gamma : [a,b] \to M$ be a normal geodesic, F be a variation of γ, we have

$$\frac{d^2}{dv^2}\Big|_{v=0} E(v) = \langle \nabla_V V, T\rangle \Big|_a^b + \int_a^b \left(\langle \nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle \right) dt$$

where V is the ~~vector~~ variational field and T is the velocity field along γ.

Generally speaking, when F is a proper variation, or $\gamma : [a,b] \to M$ is a closed geodesic, ($\Rightarrow \gamma(a)=\gamma(b), T(a)=T(b)$), we have

$$\frac{d^2}{dv^2}\Big|_{v=0} E(v) = \int_a^b \left(\underbrace{\langle \nabla_T V, \nabla_T V\rangle}_{\geq 0} - \langle R(V,T)T, V\rangle \right) dt$$

(i) If M has nonpositive curvature. $E''(0) \geq 0$
(negative) $E''(0) > 0$

$\Rightarrow$ ~~geod~~ γ is (strictly) locally minimizing.

(ii) If M has positive curvature, $-\langle R(V,T)T,V\rangle < 0$.

If $\boxed{\langle R(V,T)T,V\rangle > \langle \nabla_T V, \nabla_T V\rangle}$ (✱),

then $E''(0) < 0$, and hence γ cannot be (locally) minimizing.

The philosophy of (ii) leads to the applications of (SVF) we will discuss here soon.

Synge thm ($\Leftarrow$) When M is compact, orientable, even-dim'l, of positive curvature, for any nontrivial closed geodesic γ, we can find V s.t. (✱) holds. That is, Under the assumptions, any nontrivial (not homotopic to const curve) geodesic can not be (locally) minimizing.

Bonnet-Myers Theorem ($\Leftarrow$) When M is ~~complete and~~ of sectional curvature $\geq k > 0$, geodesics of length $> \frac{\pi}{\sqrt{k}}$ can not be (locally) minimizing.

Next, let us discuss this two applications in more detail.

$\boxed{\text{Synge Theorem}}$. [WSY, Chap. 6]. [JJ. Chap. 4., §4.1], [dC, Chap 9, §3]

Lemma 1: Let (M, g) be ~~any~~ an ~~compact~~, orientable, even-dim'l Riemannian manifold with positive sectional curvature. Then any closed geodesic which are not homotopic to a constant curve "nontrivial" for short. can not be minimizing in its (free) homotopy class.

Lemma 2: Let (M, g) be a compact Riemannian manifold. Then every (free) homotopy class of closed curves in M contains a shortest one (which is, therefore, a closed geodesic) [JJ. Theorem 1.5.1]

Remark: ① A close curve c can be considered as a continuous map $c: S^1 \to M$, where S^1 is the unit circle.

Recall that two continuous maps

$$c_0, c_1 : S^1 \to M$$

are called homotopic (同伦) if there exists a continuous map

$$F: S^1 \times [0,1] \to M$$

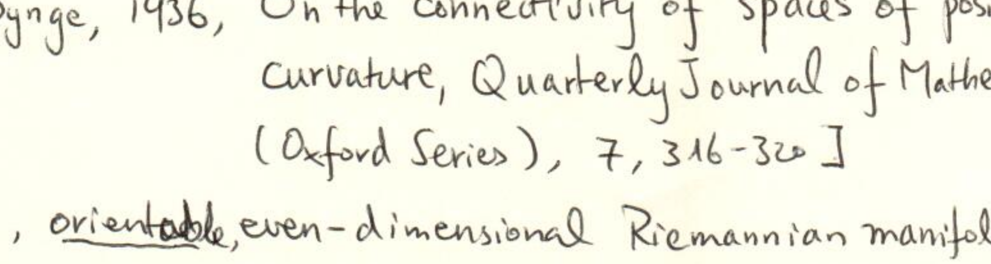
with $F(t, 0) = c_0(t)$, $F(t, 1) = c_1(t)$, $\forall\, t \in S^1$.

And the concept of homotopy defines an equivalence relation of all closed curves in M.

② Suppose (M,g) satisfy both the assumptions of Lemma 1 and that of Lemma 2, then ~~nontrivial closed geodesic doesnot~~ every homotopy class of closed curves in M contains the constant curve. That is, M is simply connected, i.e., $\pi_1(M) = \{1\}$. This is exactly what Synge Thm says.

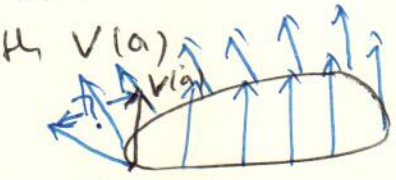
Theorem 4 [Synge, 1936, On the connectivity of spaces of positive curvature, Quarterly Journal of Mathematics (Oxford Series), 7, 316-320]
Any compact, orientable, even-dimensional Riemannian manifold with positive curvature is simply connected.

Now we start to prove Lemma 1. In fact the restrictions " orientable, even-dimensional " guarantee the existence a parallel normal vector field along a nontrivial closed geodesic



If $\gamma : [a,b] \to M$ is not a closed ~~geodesic~~ one, it is not hard to find a parallel normal vector field along it. Just pick a vector $V(a) \in T_{\gamma(a)}M$, $\langle V(a), \dot\gamma(a)\rangle = 0$ and $V(t)$ is given by the parallel ~~transform~~ transport along γ.

But for a closed one, $V(b) = P_{\gamma,a,b}V(a)$
is not necessarily coincide with $V(a)$



Note for the velocity field along a closed geodesic γ, we have

$$P_{\gamma,a,b}\,\dot\gamma(a) = \dot\gamma(b) = \dot\gamma(a)$$

That is, the orthogonal linear transformation

$$P_{\gamma,a,b} : T_pM \longrightarrow T_pM \qquad (p=\gamma(a)=\gamma(b))$$

has an eigenvalue $+1$ with eigenvector $\dot\gamma(a)$

If the multiplicity of eigenvalue $+1 \geq 2$, then we have a vector $V(p) \in T_pM$ lying in the orthogonal complement of $\dot\gamma(a)$ s.t. $P_{\gamma,a,b}V(p) = V(p)$.
Hence $V(t) := P_{\gamma,a,t}V(p)$ gives a parallel normal vector field along γ.

Next, we explain "orientable, even-dimensional" guarantee that the multiplicity of eig. $+1$ of $P_{\gamma,a,b} \geq 2$.

Since $P_{\gamma,a,b}$ is <u>orthogonal</u>, we have $\det(P_{\gamma,a,b}) = +1$ or -1.

<u>Lemma 3</u>. If $\det(P_{\gamma,a,b}) = +1$, and M is even-dim'l then the multiplicity of the eigen $+1 \geq 2$.

Proof: Since $P_{\gamma,a,b} : T_pM \to T_pM$ is orthogonal, its eigenvalues can be listed as

$$\lambda_1, \bar\lambda_1, \dots, \lambda_j, \bar\lambda_j, \underbrace{-1, \dots, -1}_{k}, \underbrace{+1, \dots, +1}_{l}.$$

where λ_i, $i=1,\dots,j$ are complex numbers with $|\lambda_i|=1$,

~~and~~

M even-dim'l $\Rightarrow$ T_pM even-dimensional

$$\Rightarrow \quad k+l \text{ is even.}$$

Since $P_{\gamma,a,b} : \dot\gamma(a) \mapsto \dot\gamma(b)=\dot\gamma(a)$, $l \geq 1$. (i.e. $l \neq 0$)

$\det(P_{\gamma,a,b}) = +1 \Rightarrow k$ even

Hence l is even and $l\neq 0$. That is $l \geq 2$. $\square$

In fact, $\det(P_{\gamma,a,b}) = +1$ is guaranteed by "orientability" of M. Let us recall briefly the concept of orientability.

Given a vector space V, let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^n$ be two basis, and $f_j = a_j^i e_i$. Then $\det(a_j^i)$ is either positive or negative. If $\det(a_j^i)>0$, we say the two basis have the same orientation. This defines an equivalence relation for all basis of V. There are exactly 2 equivalent classes. We call each of them an orientation of V.

Alternatively, the orientation of V can be described as below: Consider the dual space V^* of V. Then we have

$$\dim \Lambda^n(V^*) = 1$$

and let Ω be a basis of $\Lambda^n(V^*)$, we have

$$\Omega(f_1,\dots,f_n) = \det(a_j^i)\,\Omega(e_1,\dots,e_n).$$

That is, given a non-zero $\Omega \in \Lambda^n(V^*)$, two basis $\{e_i\}$, $\{f_j\}$ have the same orientation iff $\Omega(f_1,\dots,f_n)$ and $\Omega(e_1,\dots,e_n)$ have the same sign. In this sense, a nonzero $\Omega \in \Lambda^n(V^*)$ determines an <u>orientation</u> of V.

The second way of description can be generalized to the setting of a manifold. M is <u>orientable</u> if there exists a C^∞ nowhere vanishing n-form ω. At each $p \in M$, the basis of T_pM are divided into two classes, those with $\omega(e_1, \dots, e_n) > 0$ and those with $\omega(e_1, \dots, e_n) < 0$. The first class is called the <u>basis coherent with the orientation</u> (顺向的)

Lemma 4. Let (M, g) be an orientable Riemannian manifold and $\gamma : [a,b] \to M$ be a closed curve, Then the parallel transport $P_{\gamma,a,b} : T_pM \to T_pM$ has determinant 1.

<u>Proof</u>: Since $P_{\gamma,a,b}$ is orthogonal, we only need to show

$$\det(P_{\gamma,a,b}) > 0.$$

Let ω be a C^∞ nowhere vanishing n-form ω on M, whose existence is guaranteed by orientability. Let $\{e_i\}$ be a basis of T_pM with $\omega(e_1, \dots, e_n) > 0$.

Let $\{e_i(t)\} := \{P_{\gamma,a,t}(e_i)\}$ be the parallel transport of $\{e_i\}$ along γ. Then $t \mapsto \omega(e_1(t), \dots, e_n(t))$ is a nowhere vanishing C^∞ function on $[a,b]$. In particular, $\omega(e_1(b), \dots, e_n(b)) > 0$.

Note $\{e_i(b)\}_{i=1}^n$ is also a basis of T_pM, and

$$\omega(e_1(b), \dots, e_n(b)) = \det(P_{\gamma,a,b})\, \omega(e_1, \dots, e_n).$$

Therefore, we have

$$\det(P_{\gamma,a,b}) > 0 \qquad \square$$

Proof of lemma 1 (p. 146). ~~By lemma 3 and 4,~~ Let γ be

$\gamma : [a,b] \to M$ be a nontrivial closed geodesic in M. (let $p = \gamma(a) = \gamma(b)$)

By Lemmas 3 and 4, there exists $V(p) \in T_pM$, $\langle V(p), \dot\gamma(a) \rangle = 0$

and $P_{\gamma,a,b} V(p) = V(p)$

Therefore $V(t) := P_{\gamma,a,t} V(p)$ is a parallel normal vector field along γ.

Since $\gamma([a,b])$ is compact, there exists $\delta > 0$. s.t.

$$F : [a,b] \times (-\delta, \delta) \to M$$
$$(t, v) \mapsto \exp_{\gamma(t)} vV(t)$$

is a (geodesic) variation of γ. (existence of δ is shown by the argument we used in the proof of Corollary 1. p. 142).

Since γ is a geodesic, we have $E'(0) = 0$. Moreover,

$$E''(0) = \int_a^b \left(\langle \nabla_T V, \nabla_T V \rangle - \langle R(V,T)T, V \rangle \right) dt.$$

($\nabla_T V = 0$ since V is parallel.)

$$= -\int_a^b \langle R(V,T)T, V \rangle \, dt < 0 \text{ since sectional curvature} > 0.$$

Therefore, for $v \neq 0$ small enough, $\gamma_v : [a,b] \to M$ is a closed curve homotopic to γ but with $E(\gamma_v) < E(\gamma)$.

That is γ is not minimizing in its homotopy class.

(In fact, for length we also have

$$\ell(\gamma_v)^2 \leq 2(b-a) E(\gamma_v) < 2(b-a) E(\gamma) = \ell(\gamma)^2 \text{ since } |\dot\gamma| \equiv \text{const.} \;). \square$$

Lemma 2 is a general result for compact Rie. mfld (no curvature restriction is needed).

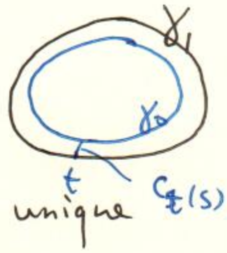
<u>Proof of Lemma 2</u> (p. 146) Recall from Corollary 3 (p. 48) of Chap. II, that for ~~the~~ a compact Rie. mfld M, there exists a $\rho_0 > 0$, s.t. any $p, q \in M$ with $d(p,q) \leq \rho_0$ can be connected by precisely one geodesic of shortest path. (Recall this is proved by using the concept of totally normal neighborhood).

~~This implies~~ Moreover, the geodesic depends continuously on (p,q). This implies immediately

<u>Claim</u>. Let (M,g) be a compact Rie. mfld, and $\rho_0 > 0$ be chosen as above. Let $\gamma_0, \gamma_1 : S^1 \to M$ be closed curves with

$$d(\gamma_0(t), \gamma_1(t)) \leq \rho_0 , \quad \forall\, t \in S^1.$$

Then γ_0, γ_1 are homotopic.



For any $t \in S^1$, let $c_t(s) : [0,1]$ be the unique shortest geodesic from $\gamma_0(t)$ to $\gamma_1(t)$. (parametrized proportionally to arclength). Since c_t depends continuously on its end points. ~~we have~~ the map

$$F(t,s) := c_t(s)$$

is continuous and yields the desired homotopy. ∎

Next we find the shortest curve in ~~the~~ a given homotopy class by method of minimizing sequence.

~~Let $\{\gamma_n\}_{n\in N}$ be a minimi~~

~~given a closed curve $c : [$~~

Let $\{\gamma_n\}_{n\in N}$ be a minimizing sequence for length in the given homotopy class. Here and in the sequel, all curves are parametrized proportionally to arc length. $\gamma_n : [0, 2\pi] \to M$.

We may assume each γ_n is piecewise geodesic. This is because: there exists m, and

$$0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=2\pi.$$

s.t. $L(\gamma_n|_{[t_{j-1},t_j]})\le \rho_0/2$.

(This realizable since one can equally divide $[0,2\pi]$ s.t. $|t_j-t_{j-1}|=\frac{\rho_0}{2|\dot\gamma|}$ $j=1,\cdots,m$. $|t_{m+1}-t_m|<\frac{\rho_0}{2|\dot\gamma|}$

and $m=\left\lceil\frac{2\pi}{\frac{\rho_0}{2|\dot\gamma|}}\right\rceil$.).

Then replacing $\gamma_n|_{[t_{j-1},t_j]}$ by the shortest geodesic arc from $\gamma_n(t_{j-1})$ to $\gamma_n(t_j)$. By the claim, this will not change the homotopy class of the curve.

Equivalently to say, we have a minimizing sequence $\{\gamma_n\}_n$ such that for each γ_n, there exists $p_{0,n},\cdots,p_{m,n}$ for which $d(p_{j-1},p_j)\le\rho_0/2$ $j=1,\cdots,m+1$ with $p_{m+1,n}=p_{0,n}$. and for which γ_n contains the shortest geodesic from p_{j-1} to p_j.

Since $\{\gamma_n\}_n$ is minimizing, ~~we may assume~~ the lengths of γ_n are bounded, say $L(\gamma_n)\le C$. Then we can assume that m is independent of n.

(This is because $L(\gamma_n)\le C\Rightarrow|\dot\gamma_n|\le\frac{C}{2\pi}\Rightarrow m\le\frac{4\pi|\dot\gamma|}{\rho_0}+1\le\frac{2C}{\rho_0}+1$)

Since M is compact, after selection of a subsequence, the points $p_{0,n},\cdots,p_{m,n}$ converge to points $p_0,\cdots,p_m$ as $n\to\infty$. The segment of γ_n from $p_{j-1,n}$ to $p_{j,n}$ then converges to the shortest geodesic from p_{j-1} to p_j (Recall such geodesic depends ~~continuou~~ continuously on its end points.).

The union of these geodesic segments yields a closed curve γ. By the Claim, γ is in ~~the~~ still in the given homotopy class and $L(\gamma) = \lim_{n\to\infty} L(\gamma_n)$, i.e. γ is the shortest one in its homotopy class. Therefore, γ has to be geodesic.

(Otherwise, there exists p and q on γ on which one of the two segments of γ from p to q has length $\leq \ell_0/2$, but is not geodesic. Then replace this segment by the unique shortest geodesic from p to q. The resulting curve lies still in the same homotopy class but with a shorter length, which is a contradiction.)

Proof of Theorem 4 (Synge). Suppose M is not simply connected. Then there is a homotopy class of closed curves which are not homotopic to a constant curve. By Lemma 2, there is a shortest closed geodesic γ in this given homotopy class. By Lemma 1, γ cannot be minimizing, this is a contradiction. $\square$

Remark ① Synge thm tells ~~that~~ any compact, orientable, evendimensional manifold which is not simply connected does not admit a metric of positive curvature.

② Examples: the real projective space $P^2(\mathbb{R})$ of dimension two, is compact, non-orientable, ~~Moreover $P^2(\mathbb{R})$ is of positive~~

Recall in Excercise 3, (2), we checked there is a Rie. metric on $P^2(\mathbb{R})$ s.t. the covering map $\pi: S^2 \to P^2(\mathbb{R})$ is a local isometry. Hence $P^2(\mathbb{R})$ has sectional curvature $1 > 0$.

But we know $\pi_1(P^2(\mathbb{R})) = \mathbb{Z}_2$. Hence "orientability" in the assumption of ~~the~~ Synge Thm is necessary.

Similarly, "even-dim'l" assumption is also necessary. $P^3(\mathbb{R})$ is orientable, compact, odd-dim'l, of positive curvature, but $\pi_1(P^3(\mathbb{R})) = \mathbb{Z}_2$. □

The above examples are inspiring and it is natural to ask what we can say when (M,g) is not orientable or not even-dimensional.

Corollary 3: Let (M,g) be a compact, non-orientable, even-dimensional Rie. mfld of positive sectional curvature, then $\pi_1(M) = \mathbb{Z}_2$.

Theorem 5 (Synge 1936) Let (M,g) be a compact, odd-dimen'l Riemannian mfld of positive sectional curvature, then M is orientable.

Remark: In particular, Corollary 3 gives a geometric proof of the fact $\pi_1(P^n(\mathbb{R})) = \mathbb{Z}_2$, when n even. (knowing $P^n(\mathbb{R})$ is non-orientable for n = even. Actually $\pi_1(P^n(\mathbb{R})) = \mathbb{Z}_2$, $\forall n$). Thm 5 gives a geometric proof of the fact $P^n(\mathbb{R})$ is orientable for n odd. But we can not say too much about the fundamental group $\pi_1(M)$ for a compact, odd-dim'l. mfld admitting a metric of positive curvature. □

The proofs ~~requires~~ use property of the orientable double cover of a non-orientable manifold. In order not to interrupt our current ~~discussion~~ topic too much, we postpone the proofs.

$\boxed{\text{Bonnet - Myers Theorem}}$: [PP, chap 6, § 4.1]

The following Lemma was first proven by Bonnet for surfaces and later by Synge for general Riemannian manifolds as an application of his (SVF).

<u>Lemma 5</u>. (Bonnet 1855 and Synge 1926). Let (M,g) be a Rie. mfld with sectional curvature $\geq k > 0$. Then geodesics of length $> \frac{\pi}{\sqrt{k}}$ cannot be (locally) minimizing.

<u>Proof</u>: Let $\gamma : [0, l] \to M$ be a normal (i.e. $|\dot\gamma|=1$) geodesic of length $l > \frac{\pi}{\sqrt{k}}$.

Let $E(0)$ be a unit vector in $T_{\gamma(0)}M$ with $\langle E(0), \dot\gamma(0)\rangle = 0$.

Then we obtain $E(t) := P_{\gamma, 0, t}\, E(0)$. a parallel vector field along γ.

Consider the following vector field along γ

$$V(t) := \sin\left(\frac{\pi}{l} t\right) E(t).$$

It corresponds to a proper variation since $V(0) = V(l) = 0$.

By (SVF):

$$\left.\frac{d^2}{du^2}\right|_{u=0} E(u) = E''(0) = \int_0^l \left(\langle \nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle\right) dt$$

Observe $\nabla_T V = \sin'\left(\frac{\pi}{l}t\right) E(t) = \frac{\pi}{l}\cos\left(\frac{\pi}{l}t\right) E(t)$

and hence $\langle \nabla_T V, \nabla_T V\rangle = \left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi}{l}t\right)$

and $\langle R(V,T)T,V\rangle = \sin^2\left(\frac{\pi}{\ell}t\right) \langle R(E,T)T,E\rangle$

$$= \sin^2\left(\tfrac{\pi}{\ell}t\right) K(E,T)$$

$$\Rightarrow E''(0) = \int_0^\ell \left(\tfrac{\pi}{\ell}\right)^2 \cos^2\left(\tfrac{\pi}{\ell}t\right) dt - \int_0^\ell \sin^2\left(\tfrac{\pi}{\ell}t\right) K(E,T)\, dt$$

$$\leq \left(\tfrac{\pi}{\ell}\right)^2 \int_0^\ell \cos^2\left(\tfrac{\pi}{\ell}t\right) dt - k \int_0^\ell \sin^2\left(\tfrac{\pi}{\ell}t\right) dt$$

$\ell > \frac{\pi}{\sqrt{k}} \Rightarrow \left(\frac{\pi}{\ell}\right)^2 < k$

$$< k \int_0^\ell \left[\cos^2\left(\tfrac{\pi}{\ell}t\right) - \sin^2\left(\tfrac{\pi}{\ell}t\right)\right] dt$$

$$= k \int_0^\ell \cos\left(\tfrac{2\pi}{\ell}t\right) dt = 0 .$$

Hence all nearby curves in the variation are shorter than γ. (by the same argument as in the end of the proof for Lemma 1 p.151) $\square$

In the above, we ~~say~~ see that we actually has $(n-1)$ choices of the parallel normal vector fields along γ. ~~In fact, the~~ When sectional curvature $\geq k > 0$, our above argument works for each of those $(n-1)$ parallel vector fields along γ. On the other hand, for our purpose here, ~~we only need~~ it's enough to know that our above argument works for at least one of those $(n-1)$ vector fields along γ. This leads to the following extension due to Myers.

Lemma 5' (Myers 1941). Let (M,g) be a Rie. mfld with Ricci curvature $Ric \geq (n-1)k > 0$. Then geodesics of length $> \frac{\pi}{\sqrt{k}}$ cannot be minimizing.

<u>Proof</u>. Similarly as in the proof of Lemma 5, let $\gamma : [0,\ell] \to M$

be ~~the~~ a normal geodesic with $l > \frac{\pi}{\sqrt{k}}$.

Choose $E_2, \cdots, E_n \in T_{\gamma(0)}M$ s.t. $\dot\gamma(0), E_2, \cdots, E_n$ form an orthonormal basis for $T_{\gamma(0)}M$. Then $E_i(t) := P_{\gamma,0,t} E_i$ and $\dot\gamma(t)$ form an orthonormal basis for $T_{\gamma(t)}M$.

Consider $n-1$ variational fields along γ

$$V_i(t) = \sin\left(\frac{\pi}{l}t\right) E_i(t), \quad i = 2, \cdots, n.$$

We have for each i,

$$\left.\frac{d^2}{dv_i^2}\right|_{v_i=0} E(v_i) = \left(\frac{\pi}{l}\right)^2 \int_0^l \cos^2\left(\frac{\pi}{l}t\right)dt - \int_0^l \sin^2\left(\frac{\pi}{l}t\right) K(E_i, T)\,dt$$

$$< k \int_0^l \cos^2\left(\frac{\pi}{l}t\right)dt - \int_0^l \sin^2\left(\frac{\pi}{l}t\right) K(E_i, T)\,dt$$

Taking the summation,

$$\sum_{i=2}^{n} \left.\frac{d^2}{dv_i^2}\right|_{v_i=0} E(v_i) < (n-1)k \int_0^l \cos^2\left(\frac{\pi}{l}t\right)dt - \int_0^l \sin^2\left(\frac{\pi}{l}t\right) \underbrace{\sum_i K(E_i, T)}_{Ric(T)}\,dt.$$

$$\leq (n-1)k \int_0^l \cos^2\left(\frac{\pi}{l}t\right)dt - (n-1)k \int_0^l \sin^2\left(\frac{\pi}{l}t\right)dt$$

$$= 0.$$

Hence there exists an $i_0 \in \{2, \cdots, n\}$ s.t.

$$\left.\frac{d^2}{dv_{i_0}^2}\right|_{v_{i_0}=0} E(v_{i_0}) < 0.$$

And hence γ is not (locally) minimizing. □

If we assume further that (M,g) is <u>complete</u>, the above lemma implies and upper bound of the diameter of (M,g). This seems to have first been pointed out by Hopf-Rinow (1931) for surfaces in their famous paper on completeness and then by Myers for general Riemannian manifolds.

(1935. Duke J. Math ~~on~~ for sectional curvature restriction
1941. Duke J. Math for Ricci curvature restriction)

<u>Corollary 4</u>. Suppose (M, g) is a <u>complete</u> Riemannian manifold with Ricci curvature $Ric \geq (n-1)k > 0$. Then

$$\mathrm{diam}(M, g) \leq \frac{\pi}{\sqrt{k}}.$$

Furthermore, (M, g) has finite fundamental group.

<u>Remark</u>: Coro. 4. is oftenly referred to as <u>Bonnet-Myers</u> Theorem.

<u>Proof</u>: Lemma 5' tells no geodesic ~~of length $\geq \frac{\pi}{\sqrt{k}}$~~ can realize distance between ~~betwe~~ any $p, q \in M$ with $d(p,q) > \frac{\pi}{\sqrt{k}}$. Hopf-Rinow ~~ta~~ Thm tells that <u>completeness</u> implies any $p, q \in M$ can be connected by a shortest geodesic. Hence

$$d(p,q) \leq \frac{\pi}{\sqrt{k}}, \ \forall p, q \in M. \qquad \square$$

Lecture 15. 2017. 04. 13

Covering spaces and Fundamental groups.

A continuous map $\pi: X \to M$
is called a <u>covering map</u> if each $p \in M$ has a neighborhood U with the property that each connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U.

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<u>FACT 1</u>: Let M be diff. mfld, X has a natural differentiable structure s.t. $\pi: X \to M$ is C^∞ and locally diffeomorphism.

Let $\{(U_\alpha, x_\alpha)\}$ be a differentiable structure of M. U_α small s.t. $\pi^{-1}(U_\alpha)$ are disjoint opensets of X. each connected component $U_\alpha^i \subset X$ we assign coordinate map $x_\alpha \circ \pi$. (note $\pi: U_\alpha^i \to U_\alpha$ is homeomorphic)

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[d, "\pi"'] \arrow[dr, "x_\alpha \circ \pi"] & \\ U_\alpha \arrow[r, "x_\alpha"'] & \mathbb{R}^n \end{tikzcd}$$

This leads to a differentiable structure for X, ~~and π~~ under which π is C^∞ and locally diffeomorphism.

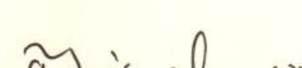
<u>FACT2</u>. Let (M, g) be a Riemannian manifold. Note π is surjective. We can assign by $\tilde{g} = \pi^* g$ a Riemannian metric for X. Then π : $(X, \tilde{g}) \to (M, g)$ is a locally isometry.

<u>FACT3</u>. If (M, g) is <u>complete</u>, then $(X, \tilde{g})$ is also <u>complete</u>.

<u>Proof</u>: ~~Let~~ Suppose γ be a normal geodesic on $(X, \tilde{g})$ with the maximal interval $[0, b)$, $b < \infty$.

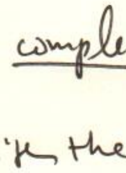
Since π is locally isometry, we have $\pi(\gamma) : [0, b) \to M$ is a geodesic of (M, g). Since (M, g) is complete, we have the geodesic $\pi(\gamma)$ in (M, g) can be extended to

$$\pi(\gamma)(b) := p \in M.$$



Pick a small normal neighborhood U of p in M,

then $\exists\ a < b$ s.t. $\pi(\gamma)(a) \in U$, and $\pi(\gamma)|_{[a,b]} \subset U$.

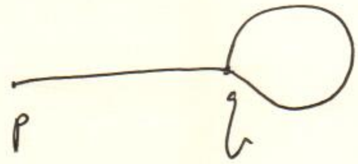


let U_i be the connected component ~~contain~~ of $\pi^{-1}(U)$ containing $\gamma(a)$.

Then the isometry $\pi^{-1} : U \to U_i$ maps a geodesic to a geodesic. Hence the geodesic γ can be extended over b in U_i. This contradicts to the maximality of b. □

The equivalence or homotopy classes of ~~loops~~ closed curves with fixed base point $p \in M$ form a group $\pi_1(M, p)$, the <u>fundamental group</u> of M with base point p.

$\pi_1(M, p)$ and $\pi_1(M, q)$ are isomorphic for any $p, q \in M$.



Hence, it make sense to speak of the fundamental group $\pi_1(M)$ of M without reference to a base point.

Let $\pi : X \to M$ be a covering map.
A <u>deck transformation</u> is a homeomorphism $\varphi : X \to X$ with

$$\pi = \pi \circ \varphi$$

<u>FACT4</u>. A deck transformation φ of $(X, \tilde{g})$ is an isometry.

<u>Proof</u>: Since π is locally isometry, and $\pi = \pi \circ \varphi$, we know φ is locally isometry. Since φ is homeomorphic. we have φ is an isometry.

~~Let $\pi(x_0) = p_0 \in M$, then π_1~~

A ~~covering~~ deck transformation which has a fixed point is the identity.

If $\pi : \tilde{M} \to M$ be the universal covering of M.

$\pi(x_0) = p_0 \in M$.

- $\pi_1(M, p_0)$ ~~biject~~ is in 1-1 correspondence with $\pi^{-1}(p_0)$.

  $x_1 \in \pi^{-1}(p_0)$ corresponds to the homotopy class of $\pi(\gamma_{x_1})$ where $\gamma_{x_1}(0) = x_0$, $\gamma_{x_1}(1) = x_1$.

- The set $\mathcal{D}$ of all deck transformations is in 1-1 correspondence with $\pi_1(M, p_0)$.

  Associate each deck transformation φ with $\varphi(p_0) \in \pi^{-1}(p_0)$.

So much for the general facts. Let's come back to our discussion about Bonnet-Meyers Thm and Synge Thm.

We have shown.

<u>Corollary 4</u>. Suppose (M, g) be a <u>complete</u> Rie. mfld with $Ric \geq (n-1)k > 0$. Then

$$\operatorname{diam}(M, g) \leq \frac{\pi}{\sqrt{k}},$$

and hence, in particular, (M, g) is compact.

(The last assertion follows from Hopf-Rinow. (The whole manifold is a bounded closed set.))

Moreover, (M, g) has finite fundamental group.

<u>Proof for the last statement</u>:

Let $\pi : \tilde{M} \to M$ be the universal covering. From our previous discussion, $(\tilde{M}, \tilde{g})$ is a C^∞ Rie. mfld, and $\pi : (\tilde{M}, \tilde{g}) \to (M, g)$ is a locally isometry. Hence the Ricci curvature of $(\tilde{M}, \tilde{g})$ is also bounded from below by $(n-1)k$.

Moreover, (M,g) is complete $\Rightarrow$ $(\tilde{M},\tilde{g})$ is complete.
Hence $\operatorname{diam}(\tilde{M},\tilde{g}) \le \frac{\pi}{\sqrt{k}}$ and $(\tilde{M},\tilde{g})$ is compact.
Then $\forall p\in M$, $\pi^{-1}(p)$ is finite. Since otherwise, $\pi^{-1}(p)$ has an accumulated point $\tilde{p}\in\tilde{M}$, and π is not a locally diffeomorphism. Therefore, the fundamental group is finite. $\square$

Extension: Cheeger-Gromoll [J.D.G. 1971] $Ric\geq 0$, positive at one point, then $\pi_1(M)$ finite.

Next, we discuss Synge theorem further.

Theorem 4. [Synge] Any compact, orientable, even-dim'l Rie. mfld. with positive curvature is simply connected.

First, recall the remarks on p. 155. that "orientable", "even-dim'l" are all necessary.

~~In contrast, the restriction "compactness" can be weakened.~~

By Bonnet-Meyers, the "compactness" can be replaced by the assumption that M is complete and has sectional curvatures bounded away from 0.

In fact, the "compactness" can be ~~weakened as~~ replaced by "completeness" alone, due to a theorem of Gromoll-Meyer.

Theorem (Gromoll-Meyer, On complete open manifolds of positive curvature, Ann. of Math., 90 (1969), 75-90)
If M is a connected, complete, non-compact n-dimensional manifold with all sectional curvatures positive, then M is diffeomorphic to $\mathbb{R}^n$.

What happens if ~~M~~ M is not orientable?

Corollary 3 (p. 155). Let (M, g) be a compact, non-orientable even-dim'l Rie. mfld of positive sectional curvature, then

$$\pi_1(M) = \mathbb{Z}_2.$$

Proof: Every non-orientable ~~di~~differential manifold M has an orientable double cover ~~of~~ $\overline{M}$:

A brief description: At each $p \in M$, the T_pM can be ~~a~~ separated into two disjoint sets: (Recall the orientation of a vector space), two bases are equivalent if ~~they~~ their transformation matrix has determinant > 0. This is an equivalence relation. Let $\mathcal{O}_p$ be the quotient space of T_pM w.r.t. the equivalence relation. $O_p \in \mathcal{O}_p$ will be called an orientation of T_pM.

$$\overline{M} = \{ (p, O_p) : p \in M,\ O_p \in \mathcal{O}_p \}.$$

Let U be a normal neighborhood of p in M. ~~with coordi-~~
choose coordinate $(x^1, \ldots, x^n)$ such that
~~Then~~

$$\left[\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right].$$

$\overline{M}$ has a natural differentiable structure s.t. $\pi : \overline{M} \to M$ is C^∞ and surjective. $\forall p \in M$, $\exists\, U \subseteq M$, $p \in U$ st.

$$\pi^{-1}(U) = V_1 \sqcup V_2.$$

$\pi : V_i \to U$ is diffeomorphism. [dC. Chap 0, Ex. 12]

Example: S^2 is the orientable double cover of $P^2(\mathbb{R})$.

Then by our previous discussion, $\bar{M}$ is orientable, and compact, even-dim'l. and positive sectional curvature. Hence $\bar{M}$ is simply connected. Therefore $\pi_1(M) = \mathbb{Z}_2$. $\square$

<u>What happens if not "even-dimensional"?</u>

<u>Thm5</u> . p. 155.

For that purpose, we prove a more general result.

<u>Theorem</u> (Weinstein, 1968). Let f be an isometry of a compact orientable Riemannian manifold M^n. suppose that M has positive sectional curvature, and f reverses the orientation of M and n is odd. Then f has a fixed point.

<u>Proof</u>: Suppose, to the contrary, ~~$f(p) \neq p$ for all $p \in M$.~~ $f(q) \neq q$ for all $q \in M$.

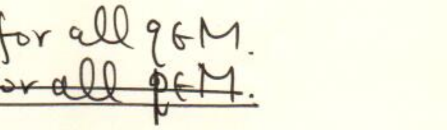
Let $p \in M$ such that $d(p, f(p))$ attains the minimum

$$d(p, f(p)) = \inf_{q \in M} d(q, f(q)) \quad (\text{We use } M \text{ is cpt})$$

Since M is cpt $\Rightarrow$ complete, $\exists$ a normal minimizing geodesic

$\gamma : [0, l] \to M$, $\gamma(0) = p$, $\gamma(l) = f(p)$

and $l = d(p, f(p))$.

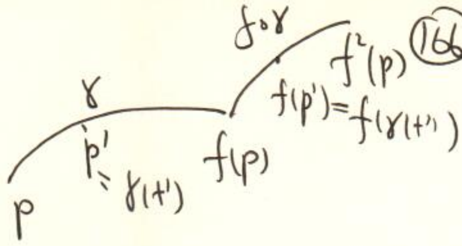


<u>Claim</u>. $(f \circ \gamma)^{\boldsymbol{\cdot}}(0) = \dot{\gamma}(l)$

Proof: Let $p' = \gamma(t')$, $t' \neq 0$, $t' \neq l$,

$f(p') = f \circ \gamma(t')$.

We have



$$d(p', f(p')) \overset{(*)}{\leq} d(p', f(p)) + d(f(p), f(p'))$$
$$\overset{f \text{ isometry}}{=} d(p', f(p)) + d(p, p')$$
$$\overset{\gamma \text{ is minimizing}}{=} d(p, f(p))$$

Then by $d(p, f(p)) = \inf_{q \in M} d(q, f(q))$, we know "$\overset{(*)}{\leq}$" is an "$=$".

i.e. $d(p', f(p')) = d(p', f(p)) + d(f(p), f(p'))$

That is the curve $\gamma|_{[t', l]} \cup f\circ\gamma|_{[0, t']}$ is a shortest curve and hence a geodesic.

In particular, this implies $(\dot{f\circ\gamma})(0) = \dot\gamma(l)$.

Next consider $P_{\gamma,0,l}^{-1} \circ df_p : T_pM \to T_pM$.

Then it is an isometry and hence, an orthogonal transformation.

Note $df_p(\dot\gamma(0)) = (\dot{f\circ\gamma})(0)$ (since $f(p) = f\circ\gamma(0)$).

we have
$$\left(P_{\gamma,0,l}^{-1} \circ df_p\right)(\dot\gamma(0)) = P_{\gamma,0,l}^{-1}\left((\dot{f\circ\gamma})(0)\right)$$
$$= P_{\gamma,0,l}^{-1}(\dot\gamma(l)) = \dot\gamma(0).$$

That is $P_{\gamma,0,l}^{-1} \circ df_p$ has eigenvalue $+1$ with multiplicity ≥ 1.

Since $P_{\gamma,0,t}$ preserves orientation and f reverse the orientation, we have
$$\det\left(P_{\gamma,0,l}^{-1} \circ df_p\right) = -1.$$

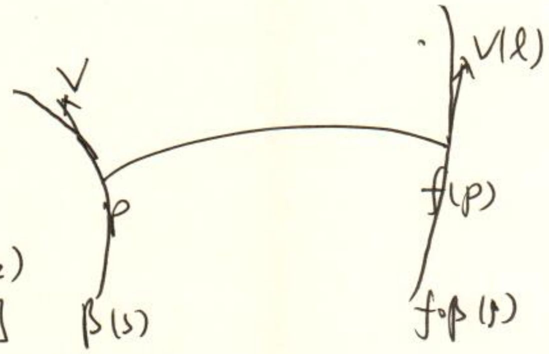
List all its eigenvalues as
$$\lambda_1, \bar\lambda_1, \cdots, \lambda_j, \bar\lambda_j, \underbrace{-1, \cdots, -1}_{l}, \underbrace{+1, \cdots, +1}_{k}.$$

We have

$$\left.\begin{array}{l} n \text{ odd} \Rightarrow k+l \text{ odd} \\ \det = -1 \Rightarrow l \text{ odd} \end{array}\right\} \Rightarrow \left.\begin{array}{l} k \text{ even} \\ k \geq 1 \end{array}\right\} \Rightarrow k \geq 2.$$

$\Rightarrow \exists\, V \in T_pM$, $\langle V, \dot\gamma(0)\rangle = 0$, and $\left(P_{\gamma,0,l}^{-1} \circ df_p\right)(V) = V$.

i.e. $P_{\gamma,0,l} V = df_p(V)$.



Define $V(t) = P_{\gamma,0,t} V$,

we have

$$F(t,s) = \exp_{\gamma(t)}(sV(t)),\ s \in (-\varepsilon, \varepsilon),\ t \in [0,l]$$

is a variation of γ,

with
$$F(t,0) = \gamma(t)$$
$$F(0,s) = \beta(s)$$
$$F(l,s) = f \circ \beta(s)$$

and $\left.\dfrac{\partial F}{\partial s}\right|_{s=0} = V(t)$.

By the (SVF), we have

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(s) = \int_a^b \langle \nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle \, dt < 0.$$

This shows that $\exists$ small enough s, s.t. the curve γ_s has the property $L(\gamma_s)^2 \leq 2l\, E(\gamma_s) < 2l\, E(\gamma) = L(\gamma)^2$.

Hence let $p_s = \gamma_s(0)$, we have

$$d(p_s, f(p_s)) \leq L(\gamma_s) < L(\gamma) = d(p, f(p))$$

which contradicts to the minimality of $d(p, f(p))$. $\square$

<u>Proof of Thm 5. p.155</u> Suppose M is not orientable. Let $\bar M$ be the orientable double cover of M. Then $(\bar M, \pi^* g)$ is a compact orientable

manifold with positive sectional curvature. Let φ be a deck transformation of $\tilde{M}$ with $\varphi \neq id$.

Because M is not orientable, φ is an isometry which reverse the orientation of $\tilde{M}$. Since n is odd, we can apply Weinstein's theorem to conclude φ has a fixed point. Therefore $\varphi = id$, which is a contradiction. $\square$

<u>Exercise</u>

(1) Prove Weinstein theorem for even-dim'l case: Let f be an isometry of a compact orientable Rie. mfld M^n. Suppose M has positive sectional curvature, and f preserve the orientation of M and n is even. Then f has a fixed point.

(2) Prove Synge Thm (even-dim'l) as a Corollary.