(V) Space forms and Jacobi fields

We start our further investigation on geometry and topology of Riemannian manifolds by studying the simplest cases: complete Riemannian manifolds with constant sectional curvature, which are called space forms. We again will study the behavior of geodesics in order to ~~know~~ reveal the underlying geometry and topology.

~~§1. Space forms.~~ The first problem we're concerned about space forms is the existence.

Recall if a Rie. mfld (M, g) has constant sectional curvature k, then $(M, \lambda g)$ has constant sectional curvature $\frac{k}{\lambda}$ for $\lambda > 0$. Therefore, we only need to consider space forms with sectional curvature $0, +1, -1$.

Obviously, $\mathbb{R}^n$ with the Euclidean metric has 0 sectional curvature. (For example, by the ~~...~~ formula in local coordinate:

$$\left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right\rangle = \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}\right)$$
$$+ g_{mp}\left(\Gamma_{il}^m \Gamma_{jk}^p - \Gamma_{jl}^m \Gamma_{ik}^p\right)$$

recalled from p.122. For $\mathbb{R}^n$, $R_{kl;ij} = 0$, $\forall\, i,j,k,l.$), Hence $\mathbb{R}^n$ is a space form with sectional curvature 0.

In fact, we have the following result:

Theorem 1. For any $c \in \mathbb{R}$ and any $n \in \mathbb{Z}^+$, there exists a unique (upto isometries) simply connected n-dimensional space form with constant sectional curvature c.

In order to discuss the existence for the other two cases $c = +1$ or -1, we first introduce some useful ideas.

§1 Isometries and totally geodesic submanifold.

Let (M, g), $(\bar{M}, \bar{g})$ be two Riemannian manifolds, and

$$f: M \longrightarrow \bar{M}$$

be an immersion (浸入). If $f^* \bar{g} = g$, the we say f is an isometric immersion, and M is called the Riemannian immersed submanifold, or submanifold for short. If f is an embedding, M is called the Riemannian embedded submfld., or regular submanifold. ~~for~~ (正则子流形)

Let $\dim M = n$, $\dim \bar{M} = n+k$, we say M has codimension k in M. In particular, if $k=1$, M is called a hypersurface in $\bar{M}$.

Definition (totally geodesic submanifold). Let M be a submanifold of $\bar{M}$. We identify $p \in M$ with $f(p) \in \bar{M}$. Then

$$T_p\bar{M} = T_pM \oplus T_p^{\perp}M.$$

where $T_p^{\perp}M$ is the orthogonal complement of T_pM in $T_p\bar{M}$.

M is called a totally geodesic submanifold if $\forall$ geodesic γ in $\bar{M}$ with $\gamma(0) \in M$, $\dot{\gamma}(0) \in T_pM$, we have $\gamma \subset M$.

Remark. Recall from the Finial Remark (p. 94) of our discussions about Levi-Civita connection, we know for the Levi-Civia connection $\bar{\nabla}$ and ∇ for $\bar{M}$ and M respectively we have

$$\bar{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0 \Rightarrow \nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

That is, γ is also a geodesic in M.

There is a characterization of totally geodesic submanifold by the <u>second fundamental form</u>.

In fact, the decomposition

$$T_p\bar{M} = T_pM \oplus T_p^{\perp}M$$

is differentiable, and, hence, the tangent bundle

$$T\bar{M} = TM \oplus NM$$

where NM is the normal bundle.

For any $X, Y \in \Gamma(TM)$, define

$$B(X,Y) = \bar{\nabla}_X Y - \nabla_X Y \in \Gamma(NM)$$

First observe. $\forall$ function f on M, we have

$$\left.\begin{aligned} B(fX, Y) &= f B(X,Y) \quad (\text{easy}) \\ B(X, fY) &= f B(X,Y) \end{aligned}\right\} (*)$$

~~Hence we~~ We also have $B(X,Y) = B(Y,X)$ (Using torsion-free property)

B is called the <u>second fundamental form</u> of the submanifold M in $\bar{M}$.

<u>Theorem 2</u>. M is a totally geodesic submanifold of $\bar{M}$ if and only if $B \equiv 0$.

<u>Proof</u>: Due to the property $(*)$, we can speak of the map $\forall p \in M$, $B : T_pM \times T_pM \to N_pM$. which is bilinear, & symmetric.

Let M be a <u>totally geodesic submanifold</u> of $\bar{M}$, then $\forall V \in T_pM$, Let γ be the geodesic in $\bar{M}$ with $\gamma(0)=p$, $\dot{\gamma}(0)=V$.

~~then~~ $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$, then we have $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$

that is $\quad \bar{\nabla}_{\dot{\gamma}}\dot{\gamma} - \nabla_{\dot{\gamma}}\dot{\gamma} = \bar{\nabla}_v v - \nabla_v v = B(v,v) = 0$

Since B is bilinear and symmetric, we have

$$B(v,v)=0 \quad \forall v \in T_pM \Rightarrow B \equiv 0.$$

Conversely, Suppose $B\equiv 0$. Then $\forall p \in M$, $V \in T_pM$, let γ be a geodesic in $\overline{M}$ with $\gamma(0)=p$, $\dot\gamma(0)=V$.

Let ξ be the geodesic in M with $\xi(0)=p$, $\dot\xi(0)=v$.

Then $\forall t$, $\overline{\nabla}_{\dot\xi(t)}\dot\xi \underset{\substack{\uparrow \\ B\equiv 0}}{=} \nabla_{\dot\xi(t)}\dot\xi = 0$

Therefore, ξ is also a geodesic in M. Due to the uniqueness of the geodesic with initial data $\gamma(0), \dot\gamma(0)$, we conclude $\xi=\gamma$. Hence $\gamma \subset M$. $\square$

Remark. Notice that by $\overline{\nabla}_{\dot\gamma}\dot\gamma$, $\nabla_{\dot\gamma}\dot\gamma$ we are actually using the induced connection. One can check $B\equiv 0 \Rightarrow \overline{\nabla}_{\dot\xi(t)}\dot\xi = \nabla_{\dot\xi(t)}\dot\xi$ $\square$

Totally geodesic submanifold ~~is a ge~~ can be considered as a generalization of geodesics. ~~Ap~~ $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ is a totally geodesic submfld but $S^2 \subset \mathbb{R}^3$ is not.

Recall ~~Proposition 8 (p. 128) in our discussion on sectional curvature~~ we have.

Proposition 1. Let M be a totally geodesic submanifold of $\overline{M}$, Denote by K and $\overline{K}$ for their sectional curvature respectively. Then every 2-dim sections $\Pi_p \subset T_pM$ and any $p\in M$, we have

$$K(\Pi_p) = \overline{K}(\Pi_p).$$

Proof. By definition and Thm 2. $\square$

Next, we give a relation between isometry and totally geodesic submanifold. (t.g.s.)

Theorem 3. ~~Let M be a t.g.s. of $\overline{M}$~~ [WSY, p 59. 引理3] [Kl., Thm 1.10.15]

Let $f : (\overline{M}, g) \to (\overline{M}, g)$ be an isometry. Then every connected component of the fixed point set

$M = \mathrm{Fix}(f) = \{ p \in \overline{M} \mid f(p) = p \}$

is a totally geodesic submanifold.

Proof: Observe that $\mathrm{Fix}(f)$ is a closed subset: It is the preimage of the diagonal in $\overline{M} \times \overline{M}$ under the differentiable mapping

$$p \longmapsto (p, f(p)) \in M \times M.$$

Let $p \in \mathrm{Fix}(f)$. If p is not isolated, consider

$H = \{ V \in T_p\overline{M} : df_p(V) = V \}$, we have H is not empty.

Since there exist $y \in \mathrm{Fix}(f)$ which is close enough to x s.t. $\exists !$ shortest geodesic γ from x to y. Since $f(x) = x$, $f(y) = y$, we have $f(\gamma)$ is also a shortest geodesic from x to y. Hence

$f(\gamma) = \gamma$. Therefore $df(\dot{\gamma}(0)) = (f \circ \gamma)^{\cdot}(0) = \dot{\gamma}(0)$.

In fact H is linear subspace of $T_p\overline{M}$.

Let δ be small enough s.t.

$$\exp_p : B(0, \delta) \subset T_p\overline{M} \longrightarrow B_p(\delta) \subset \overline{M}.$$

is a diffeomorphism.

Claim: $\exp_p(H \cap B(0,\delta)) = M \cap B_p(\delta)$

This claim implies immediately that M is a submfld of $\overline{M}$.

Proof of the claim:

(1) $\forall q \in M \cap B_p(\delta)$

Choose $V \in B(0,\delta) \subset T_pM$, s.t. $\exp_p V = q$.

and $\gamma: [0,1] \to \bar{M}$, $\gamma(t) = \exp_p tV$ is the unique shortest geodesic. By our previous discussion, $V \in H$.

Hence $M \cap B_p(\delta) \subset \exp_p(H \cap B(0,\delta))$

(2). let $V \in H \cap B(0,\delta)$, let $q = \exp_p V$.

Let $\gamma: [0,1] \to \bar{M}$ be the geodesic $\gamma(t) = \exp_p tV$.

then $\gamma(0) = p$, $\gamma(1) = q$.

then $f \circ \gamma$ is also a geodesic. with $(f\circ\gamma)(0) = f(p) = p$.

Moreover $(\dot{f\circ\gamma})(0) = df_p(\dot\gamma(0)) = df_p(V) \underset{\substack{\uparrow \\ V\in H}}{=} V = \dot\gamma(0)$.

then by the uniquess, $f\circ\gamma = \gamma$, and in particular

$$f(q) = f\circ\gamma(1) = \gamma(1) = q.$$

Hence $\exp_p V \subset B_p(\delta) \cap M$.

i.e. $\exp_p(H \cap B(0,\delta)) = B_p(\delta) \cap M$.

This complete the proof of the claim.

The above argument (2) also tells any geodesic γ in $\bar{M}$ with $\gamma(0) \in M$, $\dot\gamma(0) \in T_{\gamma(0)}M$ ~~is also~~ satisfies $f(\gamma) = \gamma$. Hence M is a totally geodesic submanifold of M $\square$.

§2 Space forms.

We continue the discussion about the existence of space forms with sectional curvature $+1$ ~~and~~ or -1.

Example 1. $S^2 \subset \mathbb{R}^3$ (unit sphere) with the induced metric of the Euclidean metric of $\mathbb{R}^3$. Since

$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\varphi \end{cases}$$

$$g|_{S^2} = (dx^2+dy^2+dz^2)|_{S^2} = d\varphi^2 + \cos^2\varphi\, d\theta^2. \quad (r=1).$$

Then $\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \right\rangle - \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} \right\rangle^2 = \cos^2\varphi$

$$\left\langle R\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} \right\rangle \qquad \cancel{= -\frac{1}{2}(-2\cos^2\varphi)}$$

$$= \frac{1}{2}\left(\underbrace{g_{\theta\varphi,\varphi\theta}}_{=0} - \underbrace{g_{\varphi\varphi,\theta\theta}}_{=0} - g_{\theta\theta,\varphi\varphi} + \underbrace{g_{\varphi\theta,\theta\varphi}}_{=0} \right)$$

$$+ g_{mp}\left(\Gamma^m_{\varphi\theta}\Gamma^p_{\theta\varphi} - \Gamma^m_{\theta\theta}\Gamma^p_{\varphi\varphi} \right)$$

$$\left(\text{Check } \Gamma^\varphi_{\varphi\theta} = \Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0. \quad \Gamma^\theta_{\varphi\theta} = -\frac{\cos\varphi\sin\varphi}{\cos^2\varphi} \right)$$

$$= -\frac{1}{2} g_{\theta\theta,\varphi\varphi} + g_{\theta\theta}\Gamma^\theta_{\varphi\theta}\Gamma^\theta_{\theta\varphi} = \cos^2\varphi$$

$\Rightarrow$ sectional curvature $K \equiv 1$.

Proposition 2. The unit sphere $S^n \subset \mathbb{R}^{n+1}$ $(n \geq 2)$ has constant sectional curvature $+1$.

Proof. $n=2$ has been checked in Ex 1.

When $n \geq 3$. Define an isometry $f: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ as below

- $f: (x^1, x^2, x^3, x^4, \dots, x^{n+1}) = (x^1, x^2, x^3, -x^4, \dots, -x^{n+1})$

It induces an isometry $f: S^n \to S^n$.

Observe the set of fixed point of $f: S^n \to S^n = \left\{ (x^1, x^2, x^3, 0, \dots, 0) \,\middle|\, \sum_{i=1}^3 x^{i2} = 1 \right\} = S^2$.

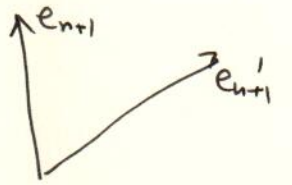
Therefore, S^2 is a totally geodesic submanifold of S^n.

Since sectional curvature of S^2 is 1, ~~so does~~ we have S^n has sectional curvature $K(\Pi_p) = 1$ for some $\Pi_p \subset T_p S^n$.

For any $\Pi'_q \subset T_q S^n$. Suppose $\Pi_p = \mathrm{span}\{e_1, e_2\}$ the positive vector of p be e_{n+1}.

$\Pi'_q = \mathrm{span}\{e'_1, e'_2\}$, the position vector of q be e'_{n+1}.

First let rotate ϕ in $\mathrm{span}\{e_{n+1}, e'_{n+1}\}$ be s.t. $\phi(e_{n+1}) = e'_{n+1}$.



and $d\phi(\Pi_p) = \Pi_q$

Then let ϕ' be the rotation ~~around~~ which fix q and send Π_q to Π'_q. Then the isometry

$$\phi' \circ \phi$$

send p to q, and $\Pi_p \subset T_p S^n$ to $\Pi'_q \subset T_q S^n$.

Hence $K(\Pi'_q) = 1$ $\square$

<u>Proposition 3</u> The unit ball

$$B^n = \{x \in \mathbb{R}^n \mid |x| < 1\} \subset \mathbb{R}^n$$

with the hyperbolic metric

$$g = \frac{4}{\left(1 - \sum_i (x^i)^2\right)^2} \sum_i dx^i \otimes dx^i$$

is a space form with constant sectional curvature -1.

<u>Proof</u>: First, we show $(B^n, g) := \mathbb{H}^n$ is complete.

Consider the curve $\gamma(s) := \left(\frac{e^s - 1}{e^s + 1}, 0, \cdots, 0 \right)$.

We compute

$$\langle \dot\gamma(s), \dot\gamma(s)\rangle_g = \frac{4}{\left(1-\left(\frac{e^s-1}{e^s+1}\right)^2\right)^2}\left(\frac{\partial}{\partial s}\left(\frac{e^s-1}{e^s+1}\right)\right)^2$$

$$= \frac{4}{\left(\frac{4e^s}{(e^s+1)^2}\right)^2}\left(\frac{e^s(e^s+1)-(e^s-1)e^s}{(e^s+1)^2}\right)^2$$

$$= \frac{(e^s+1)^4}{4e^{2s}}\cdot\frac{(2e^s)^2}{(e^s+1)^4} = 1.$$

That is γ is parametrized by arc length.

<u>Observation</u>: Any orthogonal transformation of $\mathbb{R}^n$ induces an isometry $(\mathbb{B}^n, g) \to (\mathbb{B}^n, g)$.

Let $f: \mathbb{B}^n \to \mathbb{B}^n$ be the isometry induced by

$$(x^1, x^2, \ldots, x^n) \longmapsto (x^1, -x^2, \ldots, -x^n).$$

Note $\mathrm{Fix}(f) = \gamma([0, \infty))$.

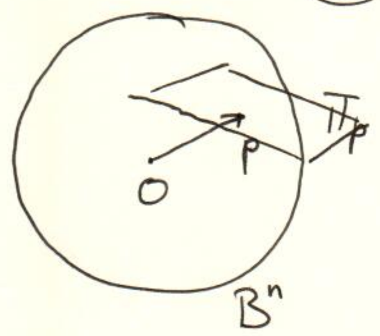
By Theorem 3 (p. 173), γ is a geodesic.

Use the <u>Observation</u> again, $A(\gamma)$ is a geodesic for any isometry A induced by ~~the~~ a orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$. That is all geodesic starting from O can be ~~extended~~ defined on $[0, \infty)$. We conclude the completeness by Hopf-Rinow Theorem.

Next, we show $\mathbb{H}^n$ has constant sectional curvature -1:

i.e. $\forall\, p \in \mathbb{B}_n$, $\forall$ 2-dim section $\Pi_p \subset T_p\mathbb{B}_n$. we have to show $K(\Pi_p) = -1$.

Let $\vec{p}$ be the position vector of p.



Identifying $T_p\mathbb{B}^n$ with $\mathbb{R}^n$.

Let E be ~~the~~ a 3-dimensional linear subspace of $\mathbb{R}^n$ containing $\vec{p}$ and Π_p.

(If $\vec{p} \in \Pi_p$, or $\vec{p} = 0$, the choice of E is not unique).

(The reason we have to consider such a 3-dim subspace: There is ~~not~~ no obvious way to say $\mathbb{H}^n$ is homogeneous, i.e. ~~$\exists$ isomet~~ $\forall\ p, q \in \mathbb{B}^n$, $\exists$ isometry $f: \mathbb{B}^n \to \mathbb{B}^n$ s.t. $f(p) = f(q)$. However, we will show this later.)

Let $\mathbb{R}^n = E \oplus E^\perp$. Let f ~~be~~ $: \mathbb{B}^n \to \mathbb{B}^n$ be the isometry induced by the orthogonal transformation

$$(e, e') \mapsto (e, -e'), \quad e \in E,\ e' \in E^\perp.$$

Observe $\mathrm{Fix}(f) = E \cap \mathbb{B}^n$.

Use the <u>Observation</u> again. Choose orthogonal transformation A s.t. $A(E) = \{(x_1, x_2, x_3, 0, \dots, 0)\} \subset \mathbb{R}^n$.

A induce an isometry $\mathbb{B}^n \to \mathbb{B}^n$.

Hence, it remains to show $\mathbb{B}^3$ with the hyperbolic metric has constant sectional curvature -1.

Use the spherical coordinate $\{\rho, \varphi, \theta\}$ on $\mathbb{B}^3 \setminus \{0\}$, the hyperbolic metric can be written as

$$\frac{4}{(1-\rho^2)^2}\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \cos^2\theta\, d\varphi^2\right)$$

where $d\rho^2 := d\rho \otimes d\rho$ and, similarly, $d\theta^2$, $d\varphi^2$.

Consider vector fields

$$X_1 = \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}, \quad X_2 = \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}, \quad X_3 = \frac{1-\rho^2}{2\rho\cos\theta}\frac{\partial}{\partial\varphi}$$

Then we have $\langle X_i, X_j\rangle = \delta_{ij}$.

We calculate $[X_1,X_2](f) = \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\frac{\partial f}{\partial\theta}\right) - \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\left(\frac{1-\rho^2}{2}\frac{\partial f}{\partial\rho}\right)$

$$= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\right)\frac{\partial f}{\partial\theta} + \frac{1-\rho^2}{2}\frac{1-\rho^2}{2\rho}\frac{\partial^2 f}{\partial\rho\partial\theta} - \frac{1-\rho^2}{2\rho}\frac{1-\rho^2}{2}\frac{\partial^2 f}{\partial\theta\partial\rho}$$

$$= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\right)\frac{\partial}{\partial\theta}f \qquad \forall f$$

$$\Rightarrow [X_1,X_2] = \frac{1-\rho^2}{2}\cdot\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\right)\frac{\partial}{\partial\theta} = \frac{1-\rho^2}{2}\,\frac{-(\rho^2+1)}{2\rho^2}\frac{\partial}{\partial\theta}$$

$$\left(\frac{\partial}{\partial\rho}\left(\frac{1-\rho^2}{2\rho}\right) = \frac{-2\rho(2\rho)-2(1-\rho^2)}{4\rho^2} = \frac{-4\rho^2-2+2\rho^2}{4\rho^2} = -\frac{\rho^2+1}{2\rho^2}\right)$$

$$\Rightarrow \quad [X_1,X_2] = -\frac{1+\rho^2}{2\rho}X_2. \qquad (1)$$

Similarly, $[X_2, X_3] = +\frac{1-\rho^2}{2\rho}\tan\theta\, X_3 \quad (2)$

$$[X_1,X_3] = -\frac{1+\rho^2}{2\rho}X_3. \qquad (3)$$

Recall for orthonormal vector fields X, Y, Z, we have by Koszul formula

$$2\langle\nabla_X Y, Z\rangle = -\langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle.$$

Employing (1), (2), and (3), we have

$$\nabla_{X_1}X_1 = \nabla_{X_1}X_2 = \nabla_{X_1}X_3 = 0$$

$$\nabla_{X_2}X_2 = -\frac{\rho^2+1}{2\rho}X_1 \qquad \nabla_{X_2}X_3 = 0$$

$$\nabla_{X_3}X_3 = -\frac{\rho^2+1}{2\rho}X_1 + \frac{1-\rho^2}{2\rho}\tan\theta\, X_2.$$

By torsion-free property ($\nabla_X Y - \nabla_Y X = [X,Y]$), the above information is enough to calculate the sectional curvature.

$$\begin{cases} K(X_1,X_2) = \langle R(X_1,X_2)X_2, X_1\rangle. \\ K(X_2,X_3) = \langle R(X_2,X_3)X_3, X_2\rangle. \\ K(X_1,X_3) = \langle R(X_1,X_3)X_3, X_1\rangle. \end{cases}$$

$$R(X_1,X_2)X_2 = \nabla_{X_1}\nabla_{X_2}X_2 - \nabla_{X_2}\underbrace{\nabla_{X_1}X_2}_{=0} - \nabla_{[X_1,X_2]}X_2$$

$$= \nabla_{X_1}\left(-\frac{\rho^2+1}{2\rho}X_1\right) - \nabla_{-\frac{1+\rho^2}{2\rho}X_2}X_2$$

$$= \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(-\frac{\rho^2+1}{2\rho}\right)X_1 + \frac{1+\rho^2}{2\rho}\cdot\left(-\frac{\rho^2+1}{2\rho}\right)X_1$$

$$\boxed{\frac{\partial}{\partial\rho}\left(\frac{\rho^2+1}{2\rho}\right) = \frac{2\rho\cdot 2\rho - 2(\rho^2+1)}{4\rho^2} = \frac{2\rho^2-2}{4\rho^2} = \frac{\rho^2-1}{2\rho^2}}$$

$$\Rightarrow R(X_1,X_2)X_2 = -\frac{1-\rho^2}{2}\cdot\frac{\rho^2-1}{2\rho^2}X_1 - \frac{(\rho^2+1)^2}{4\rho^2}X_1$$

$$= \frac{(\rho^2-1)^2 - (\rho^2+1)^2}{4\rho^2}X_1 = -X_1.$$

Hence $K(X_1,X_2) = -1$.

$$R(X_1,X_3)X_3 = \nabla_{X_1}\nabla_{X_3}X_3 - \nabla_{X_3}\underbrace{\nabla_{X_1}X_3}_{=0} - \nabla_{[X_1,X_3]}X_3$$

$$= \nabla_{X_1}\left(-\frac{\rho^2+1}{2\rho}X_1 + \frac{1-\rho^2}{2\rho}\tan\theta\, X_2\right) - \nabla_{-\frac{1+\rho^2}{2\rho}X_3}X_3$$

$$= X_1\left(-\frac{\rho^2+1}{2\rho}\right)X_1 + X_1\left(\frac{1-\rho^2}{2\rho}\tan\theta\right)X_2 + \frac{1+\rho^2}{2\rho}\nabla_{X_3}X_3$$

$$= -\frac{1-\rho^2}{2}\cdot\frac{\rho^2-1}{2\rho^2}X_1 - \cancel{\frac{1-\rho^2}{2}\frac{\rho^2+1}{2\rho^2}\tan\theta\, X_2}$$

$$- \frac{1+\rho^2}{2\rho}\frac{\rho^2+1}{2\rho}X_1 + \cancel{\frac{1+\rho^2}{2\rho}\frac{1-\rho^2}{2\rho}\tan\theta\, X_2}$$

$$= \frac{(\rho^2-1)^2 - (\rho^2+1)^2}{4\rho^2}X_1 = -X_1$$

Hence $K(X_1,X_3) = -1$.

$$R(X_2,X_3)X_3 = \nabla_{X_2}\nabla_{X_3}X_3 - \nabla_{X_3}\overbrace{\nabla_{X_2}X_3}^{=0} - \nabla_{[X_2,X_3]}X_3$$

$$= \nabla_{X_2}\left(-\frac{\rho^2+1}{2\rho}X_1 + \frac{1-\rho^2}{2\rho}\tan\theta X_2\right) - \nabla_{+\frac{1-\rho^2}{2\rho}\tan\theta X_3}X_3$$

$$= \frac{1-\rho^2}{2\rho}\left[\underbrace{\frac{\partial}{\partial\theta}\left(-\frac{\rho^2+1}{2\rho}\right)}_{=0}X_1 - \frac{\rho^2+1}{2\rho}\underbrace{\nabla_{X_2}X_1}_{=[X_2,X_1]} + \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\left(+\frac{1-\rho^2}{2\rho}\tan\theta\right)X_2\right.$$

$$\left. + \frac{1-\rho^2}{2\rho}\tan\theta\,\nabla_{X_2}X_2\right] - \frac{1-\rho^2}{2\rho}\tan\theta\,\nabla_{X_3}X_3$$

$$= -\frac{\rho^2+1}{2\rho}\cdot\frac{1+\rho^2}{2\rho}X_2 + \frac{(1-\rho^2)^2}{4\rho^2}\frac{1}{\cos^2\theta}X_2$$

$$- \frac{1-\rho^2}{2\rho}\tan\theta\,\cancel{\frac{\rho^2+1}{2\rho}X_1} + \frac{1-\rho^2}{2\rho}\cancel{\tan\theta\,\frac{\rho^2+1}{2\rho}}X_1$$

$$- \frac{1-\rho^2}{2\rho}\tan\theta\,\frac{1-\rho^2}{2\rho}\tan\theta\,X_2$$

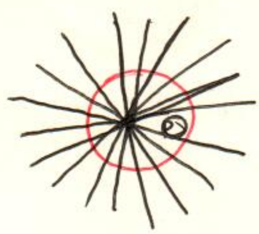
$$= -\frac{(\rho^2+1)^2}{4\rho^2}X_2 + \frac{(1-\rho^2)^2}{4\rho^2}\Big[\underbrace{\frac{1}{\cos^2\theta} - \tan^2\theta}_{=1}\Big]X_2 \qquad \square$$

$$= \frac{(\rho^2-1)^2-(\rho^2+1)^2}{4\rho^2}X_2 = -X_2.$$ Hence $K(X_2,X_3) = -1$.

Similarly, one can further check that $\langle R(X_i,X_j)X_k,X_l\rangle = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$

§3. Geodesics in $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$.

Since $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$ are totally geodesic submanifold of $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ respectively, we only need to consider geodesics in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$.



$\mathbb{R}^2$

diverge

$\mathbb{S}^2$

converge

with hyperbolic metric

$\mathbb{H}^2$

diverge

Let us measure the convergence/divergence properties of geodesic emanating from a reference point O by the length of the circle

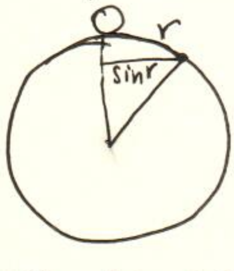
$$C(r) := \{x \in M : d(O,x) = r\}$$

When r is small enough, $C(r)$ is the image of $\mathbb{S}(0,r) \subset T_0 M$ under the diffeomorphism $\exp_0$, i.e. $C(r) = \exp_0 \mathbb{S}(0,r)$.

Let $c_0(r)$, $c_+(r)$, $c_-(r)$ be the length of $C(r)$ in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$, respectively.

(1) $\boxed{c_0(r) = 2\pi r}$. is linear in r.

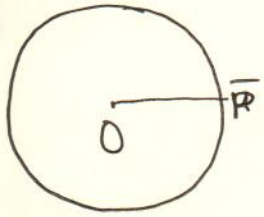
(2) $c_+(r)$, i.e. $M = \mathbb{S}^2$,



$\boxed{c_+(r) = 2\pi \sin r}$.

(Since the metric on $\mathbb{S}^2$ is induced from $\mathbb{R}^2$)

(3) $c_-(r)$, i.e. $M = \mathbb{H}^2$. (Here, we choose O to be the centre of the disc. Later, we will see $c_-(r)$ does not depend on the choice of O). Recall the normal geodesic emanating from O is given by



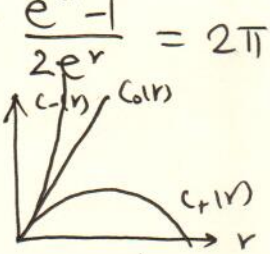
$$\gamma : [0,\infty) \to \mathbb{B}^2.$$

$$\gamma(s) = \left(\frac{e^s - 1}{e^s + 1}\right) \cdot \bar{p}, \quad \forall \bar{p} \in \partial \mathbb{B}^2.$$

Then

$$c_-(r) = \int_0^{2\pi} \frac{2}{(1-\rho^2)} \cdot \rho \, d\theta \,\Big|_{\rho = \frac{e^r - 1}{e^r + 1}}$$

$$= \frac{2 \cdot \frac{e^r - 1}{e^r + 1}}{1 - \left(\frac{e^r - 1}{e^r + 1}\right)^2} 2\pi = 2\pi \frac{e^{2r} - 1}{2e^r} = 2\pi \frac{e^r - e^{-r}}{2}$$



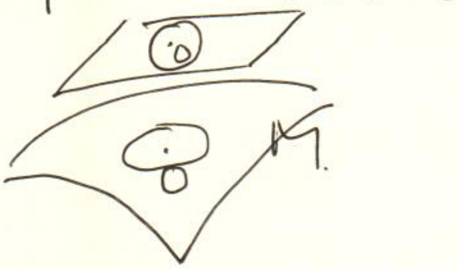
$\Rightarrow$ $\boxed{c_-(r) = 2\pi \sinh r.}$

We see that $c_-(r)$ grows much faster than $c_0(r)$.

In the above 3 particular cases, we see the <u>sign</u> of the curvature is closely related to the behavior of geodesics. <u>What happens in general</u>?

In order to answer this question, we consider the quantity $c(r)$ for a Riemannian manifold (M,g). Let $O \in M$, and $\delta > 0$ be a small number such that $\exp_O$ is a diffeomorphism on $B(0,\delta) \subset T_O M$.

Consider the ~~Riemannian~~ polar coordinate ~~around O~~ (ρ, θ) in $T_O M$.



Then for any fixed r, $\tilde{\gamma}(\theta) = (r,\theta)$ is a curve in $T_O M$. ~~Let $r < \delta$,~~

$$\frac{d}{d\theta}(r,\theta)$$

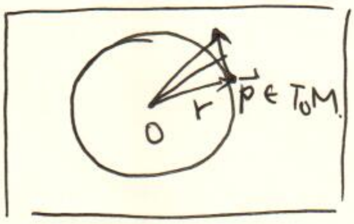
is the velocity field along $\tilde{\gamma}(\theta)$.

Let $r < \delta$, we have

$$c(r) = \int_0^{2\pi} \left\langle d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right), d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right)\right\rangle^{1/2} d\theta$$

So, for our purpose, we have to explore the interaction between the norm of $d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right)$ and the <u>curvature</u> of (M,g).

Note that, if we write



$T_O M$

$$\mathbb{R}^n \cong T_O M \ni \vec{p} = (r, \theta_p)$$

$$d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right) = d\exp_O(\vec{p}) \cdot \left(\frac{d}{d\theta}\right)$$

In order to calculate its norm, we first observe it ~~is~~ can be extended to be a variational field of a geodesic variation of

$$\gamma(t) = \exp_O \frac{t}{r}\vec{p}, \quad t \in [0, r].$$

<u>In fact</u>, we pick $F : [0,r] \times (-\varepsilon, \varepsilon) \longrightarrow M$

$$(t, s) \longmapsto \exp_O \frac{t}{r}\left(\vec{p} + s\frac{d}{d\theta}\right).$$

We observe that $F(t,0) = \gamma(t)$.

and $\frac{\partial F}{\partial s}(t,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_O \frac{t}{r}(\vec{p} + s\frac{d}{d\theta})$ is the variational field along γ. In particular

$$\frac{\partial F}{\partial s}(\cancel{t}r,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_O (\vec{p} + s\frac{d}{d\theta})$$
$$= d\exp_O(\vec{p})\left(\frac{d}{d\theta}\right)$$
$$= d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right).$$

~~Instead of ca~~. In order to calculate $\frac{\partial F}{\partial s}(r,0)$, we calculate the whole variational field. $V(t) = \frac{\partial F}{\partial s}(t,0)$, $t \in [0,r]$.

Here, we can be slightly more general: consider a general vector

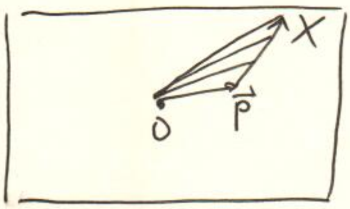
$$X \in T_{\vec{p}}(T_OM)$$

and the variation

$$F(t,s) = \exp_O \frac{t}{r}(\vec{p} + sX), \quad t \in [0,r], \ s \in (-\varepsilon,\varepsilon).$$

Let $V(t) := \frac{\partial F}{\partial s}(t,0)$ be the geodesic variational field along γ.

To calculate $V(t)$, $t \in [0,r]$, we derive the equations it satisfies: ~~Restricting to the geodesic γ, we have (let $T(t) = \gamma'(t)$.)~~



T_OM

$$\cancel{\nabla_T \nabla_T V = \nabla_T \nabla_T \frac{\partial}{\partial s} F(t,0)}$$

$$\tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} = \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t} = \tilde{\nabla}_{\frac{\partial}{\partial s}} \underbrace{\tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}}_{0 \text{ since } F \text{ is geodesic}} + \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t} - \tilde{\nabla}_{\frac{\partial}{\partial s}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}$$

$$= R\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}\right)\frac{\partial F}{\partial t}.$$

Restricting to the (normal) geodesic γ, we have

$$\nabla_T \nabla_T V = R(T,V)T, \text{ or } \nabla_T \nabla_T V + R(V,T)T = 0 \quad (*)$$

Definition (Jacobi field). Let $\gamma : [a,b] \to M$ be a geodesic, and T be the velocity field along γ. If a vector field V along γ satisfies

$$\nabla_T \nabla_T V + R(V,T)T = 0, \qquad (1)$$

we call V a Jacobi field (along γ). The equation (1) is called the Jacobi equation.

Choose parallel vector fields $Y_1, \cdots, Y_n$ along γ which are orthonormal at $\gamma(a)$, and hence orthonormal everywhere along γ. Then $\exists f^i(t)$ s.t.

$$V(t) = f^i(t) Y_i(t).$$

and $\nabla_T \nabla_T V + R(V,T)T = \frac{d^2 f^i}{dt^2} Y_i + f^i R(Y_i,T)T = 0$

$$\Leftrightarrow \left\langle \frac{d^2 f^i}{dt^2} Y_i, Y_j \right\rangle + \langle f^i R(Y_i,T)T, Y_j \rangle = 0, \ \forall j = 1, \cdots, n$$

$$\Leftrightarrow \frac{d^2 f^j}{dt^2} + f^i \underbrace{\langle R(Y_i,T)T, Y_j \rangle}_{a_i^j(t)} = 0, \ \forall j = 1, \cdots, n.$$

Hence, $V(t)$ is the solution of the above system of second order linear ODE. It will be determined by its initial conditions

$$V(0) \quad \text{and} \quad \dot V(0) := \tilde\nabla_T V(0) \in T_0 T_{\gamma(0)} M$$

where $\gamma(0) = 0$.

Recall $V(0) = \frac{\partial}{\partial s}\Big|_{s=0} F(t,s) \Big|_{t=0}$

$$= \frac{\partial}{\partial s}\Big|_{s=0} F(0,s) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_0 0 = 0.$$

$$\dot V(0) = \tilde\nabla_T V(0) = \tilde\nabla_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}\Big|_{s=0}(0,s) = \tilde\nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}(0,0)$$

$$= \tilde\nabla_{\frac{\partial}{\partial s}} \frac{1}{r}(\vec p + sX).$$

Note $\frac{1}{r}(\vec{p}+sX)$ is a vector field along the constant curve O.

$O \in M$

By definition of induced connection, we have

$$\dot{V}(0) = \tilde{\nabla}_{\frac{\partial}{\partial s}} \left(\frac{1}{r}(\vec{p}+sX)\right) = \frac{X}{r} \in T_O(T_O M)$$

i.e. $\dot{f}^i(0)\, Y_i(0) = \frac{X}{r} = \langle \frac{X}{r}, Y_i(0)\rangle Y_i(0)$

i.e. $\dot{f}^i(0) = \langle \frac{X}{r}, Y_i(0)\rangle$.

So, in order to solve $V(t)$, where $V(r) = d\exp_O(r\vec{p})\cdot(X)$ is what we need, we have to solve.

$$\begin{cases} \ddot{f}^j + \langle R(Y_i,T)T, Y_j\rangle f^i = 0, & j=1,\cdots,n \\ f^j(0) = 0 \\ \dot{f}^j(0) = \frac{1}{r}\langle X, Y_j(0)\rangle, \end{cases}$$

~~then $c(r) = \int_0^{2\pi} \langle d\exp_O(\frac{d}{d\theta}$ r.~~

Next, we come back to the calculation of $c(r)$:

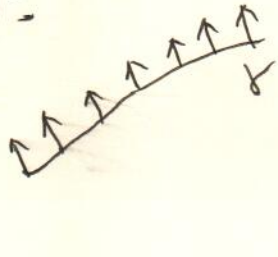
$$c(r) = \int_0^{2\pi} \langle \underbrace{d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right)}_{\substack{\parallel \\ V(r) = f^i(r)Y_i(r)}},\ d\exp_O\left(\frac{d}{d\theta}(r,\theta)\right)\rangle^{\frac{1}{2}} d\theta$$

where

$$\begin{cases} \ddot{f}^j + \langle R(Y_i,T)T, Y_j\rangle f^i = 0, \ j=1,\cdots,n. \\ f^j(0) = 0 \\ \dot{f}^j(0) = \frac{1}{r}\langle \frac{d}{d\theta}(r,\theta), Y_j(0)\rangle \end{cases} \quad (\ast)$$

In particular, for the cases of $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$.

Let $Y(t)$ be a unit parallel vector field along γ s.t. $\langle Y(t), T(t)\rangle = 0$, $\forall t$.



By Gauss' lemma, $\frac{d}{d\theta}(t,\theta)$, $t \in [0,r]$ is vertical to the radial geodesic at every t. In fact, the variational field $V(t)$ along γ is perpendicular to $\dot{\gamma}$ everywhere. (by First variation formula).

So, we can write

$$V(t) = f(t) Y(t).$$

Then the equations (#) become

$$\begin{cases} \ddot{f}(t) + \langle R(Y,T)T, Y\rangle f(t) = 0 \\ f(0) = 0 \\ \dot{f}(0) = \frac{1}{r}\langle \frac{d}{d\theta}(r,0), Y(0)\rangle = \frac{1}{r} \cdot \underbrace{\left|\frac{d}{d\theta}(r,0)\right|}_{= r} = 1 \end{cases}$$

Recall we have constant sectional curvature in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$.

Therefore, we need solve

$$\begin{cases} \ddot{f}(t) + K f(t) = 0 \\ f(0) = 0 \\ \dot{f}(0) = 1 \end{cases} \qquad K = 0, +1, \text{ or } -1 \qquad (\#\#)$$

The solution is given by

$$f(t) = \begin{cases} t, & K = 0 \\ \sin t, & K = +1 \\ \sinh t, & K = -1. \end{cases}$$

$$(\sinh' t = \cosh t, \ \cosh' t = \sinh t)$$

Therefore, we recover the results:

$$\begin{cases} C_0(r) = 2\pi r \\ C_+(r) = 2\pi \sin r \\ C_-(r) = 2\pi \sinh r \end{cases}$$

Therefore, (##) establish the relations between $c(r)$ and the curvature.

§4. What is a Jacobi field?

We have already seen the definition of Jacobi fields (p.185). Now, we want to understand this concept further.

As we have explained, it is a solution of a system of second order ODE:
$$\frac{d^2 f^j}{dt^2} + f^i \langle R(Y_i, T)T, Y_j\rangle = 0, \quad j = 1, 2, \cdots, n.$$
and $Y_1, \cdots, Y_n$ are parallel orthonormal vector fields along γ.

And the Jacobi field is given by
$$V(t) = f^i(t)\, Y_i(t).$$

<u>Proposition 4</u> Let $\gamma : [a,b] \to M$ be any geodesic.

(1) Given $V, W \in T_{\gamma(a)}M$, there exists a unique Jacobi field $U(t)$, $t \in [a,b]$ such that $U(0) = V$, $\tilde{\nabla}_{\frac{\partial}{\partial t}} U(t) := \dot{U}(0) = W$.

(2) The linear space of all Jacobi fields along γ is of $2n$ dim'l.

(3) The zero points of a Jacobi field U along γ are <u>discrete</u>, if U is not identically 0 along γ.

<u>Proof</u>: (1), (2) follow directly from the theory of 2nd linear ODEs.

Given $U(0)$, $\dot{U}(0)$, the 2nd order linear ODE has a unique solution.

For (3), assume the zero points are not distscrete. Then there is an accumulated point $\gamma(t_0)$. Then $U(t_0) = 0$, and
$$\dot{U}(t_0) = \tilde{\nabla}_{\frac{\partial}{\partial t}}\left(f^i(t)\frac{\partial}{\partial x^i}\right) = \frac{df^i}{dt}(t_0)\frac{\partial}{\partial x^i} + f^i(t_0)\,\tilde{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x^i}(t_0).$$
pick (x^i) to be the normal coordinate around $\gamma(t_0)$, then
$$\dot{U}(t_0) = \frac{df^i}{dt}(t_0)\frac{\partial}{\partial x^i} = 0.$$
Then U is identically zero along γ. $\square$

From our discussions in §3. (p.184), we have seen that the variational field of a geodesic variation of a geodesic γ is a Jacobi field along γ. In fact, the converse also hold.

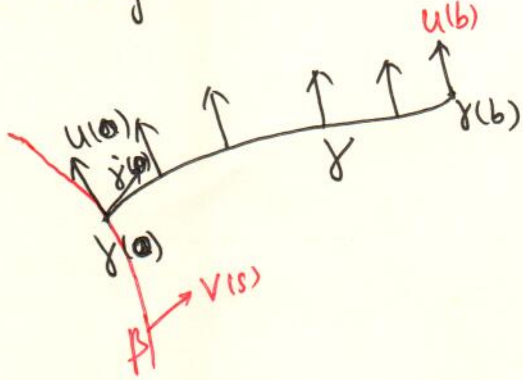
<u>Proposition 5</u>: Let $\gamma : [0,b] \to M$ be a geodesic and U be a vector field along γ. Then U is a Jacobi field <u>if and only if</u> U is the variational field of a geodesic variation of γ.

<u>Proof</u>: ($\Leftarrow$) The calculations in the end of p.184 proves this direction.

($\Rightarrow$) Let U be a Jacobi field along γ.

Let $\beta : (-\varepsilon, \varepsilon) \to M$ be the geodesic with

$$\begin{cases} \beta(0) = \gamma(0) \\ \dot\beta(0) = U(0) \end{cases}$$



We put
$$F : [0,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$$
$$(t,s) \longmapsto \exp_{\beta(s)} t\left(V(s) + sW(s)\right).$$

where V, W are parallel vector fields along β with

$$V(0) = \dot\gamma(0), \quad W(0) = \dot U(0) = \tilde\nabla_{\frac{\partial}{\partial t}} U(0).$$

Then $F(t,0) = \exp_{\beta(0)} tV(0) = \exp_{\gamma(0)} t\dot\gamma(0) = \gamma$.

and $F(t,s) = \exp_{\beta(s)} t(V(s) + sW(s))$ are all geodesics for $s \in (-\varepsilon, \varepsilon)$

That is F is a geodesic variation of γ. Therefore its variational field

$$Y(t) =: \frac{\partial F}{\partial s}(t,0) = \frac{\partial F}{\partial s}(t,s)\Big|_{s=0}$$

is a Jacobi field. Meanwhile, we have

$$Y(0) = \frac{\partial}{\partial s}\Big|_{s=0} F(0,s) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\beta(s)} 0 = \dot\beta(0) = U(0)$$

and
$$\dot Y(0) = \tilde\nabla_{\frac{\partial}{\partial t}} \cdot \frac{\partial}{\partial s} F(t,s)\Big|_{s=0}$$

$$= \tilde\nabla_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}(t,s)\Big|_{s=0} \qquad \text{(torsion free)}$$

$$= \tilde\nabla_{\frac{\partial}{\partial s}} \underbrace{(V(s)+sW(s))}_{\text{a v.f. along the constant curve } \gamma(0)=\beta(s)}\Big|_{s=0}$$

$$= W(0) = \dot U(0).$$

Then Proposition 4 (1) (p.188) implies that $U = Y$. That is U is the variational field of F. $\square$

Remark: We summarize what we learned about Jacobi fields up to now:

(1) Let $\beta: (-\varepsilon, \varepsilon) \to M$ be a curve, $V(s)$, $W(s)$ are parallel vector fields along β. Then the family of geodesics

$$\gamma_s(t) := \exp_{\beta(s)} t(V(s)+sW(s))$$

leads to a geodesic variation $F(t,s) := \gamma_s(t)$ ~~that~~ whose variational field along $\gamma_0(t)$ is a Jacobi field $U(t)$ with

$$U(0) = \dot\beta(0), \quad \dot U(0) = W(0).$$

In particular, when $\beta(s) = p \in M$ is a constant curve, we have

(2) The 1-parameter family of geodesics

$$\gamma_s(t) = \exp_p t(V + sW), \quad V, W \in T_pM.$$

gives the Jacobi field $U(t)$ along γ_0 with

$$U(0) = \dot\beta(0) = 0, \quad \dot U(0) = W.$$

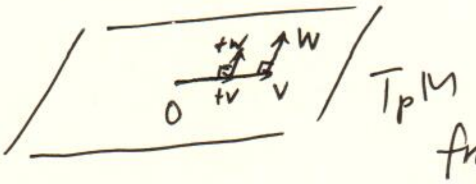
Since T_pM is an inner product space, we can restrict $\langle V, W\rangle = 0$. Then, we have.

(3). The 1-parameter family of geodesics

$$\gamma_s(t) = \exp_p t(V+sW), \quad \langle V, W\rangle = 0.$$

gives a "$\underbrace{\text{normal Jacobi field}}_{\Downarrow}$" $U(t)$, with $U(0)=0$, $\dot{U}(0)=W$.

$$\langle U(t), \dot\gamma_0(t)\rangle = 0, \ \forall t.$$



Observe $\langle tW, tV\rangle = 0$. Recall the Gauss lemma we derived from the First variation formula, we have, since $U(t)$ is the variational field, $\langle U(t), \dot\gamma_0(t)\rangle = 0$. □

We will see later that normal Jacobi fields with $U(0)=0$ are very important for our further investigation.

Relations with the SVF. Recall for proper variations of a geodesic γ, we have

$$\frac{\partial^2}{\partial v\,\partial w}\Big|_{(0,0)} E(v,w) = \underset{\text{the index form}}{I(V,W)} = \int_a^b \langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle\, dt$$

Observe that $I(V,W)$ is bilinear. $I(V,W)$ is also symmetric since E is C^∞ in (v,w).

Returning to the original problem: to determine whether a geodesic is (locally) minimizing. For that purpose, we hope to decide whether $\det\left(\frac{\partial^2}{\partial v\,\partial w}\Big|_{(0,0)} E(v,w)\right)$ is positive or not. We will see the existance of Jacobi field (vanishing at the two ends $\gamma(a)$, $\gamma(b)$) will be an obstruction.

Proposition 6. Let $\gamma:[a,b]\to M$ be a geodesic and U be a vector field along γ. Then U is a Jacobi field if and only if

$$I(U,Y) = 0$$

for all vector fields Y along γ with $Y(a) = Y(b) = 0$.

Proof: $I(U,Y) = \int_a^b \langle \nabla_T U, \nabla_T Y\rangle - \langle R(U,T)T, Y\rangle \, dt$

$$= \int_a^b \langle -\nabla_T\nabla_T U, Y\rangle - \langle R(U,T)T, Y\rangle \, dt$$

since ∇ is compatible with g and $Y(a) = Y(b) = 0$

$$= \int_a^b \langle -\nabla_T\nabla_T U - R(U,T)T, Y\rangle \, dt.$$

for all Y with $Y(a) = Y(b) = 0$.

Therefore $\nabla_T\nabla_T U + R(U,T)T = 0$ holds by the fundamental lemma of the calculus of variations. □

Proposition 7. Let $\gamma : [a,b] \to M$ be a geodesic and U be a vector field along γ. Then U is a Jacobi field <u>if and only if</u> it is a critical point of $I(X,X)$ w.r.t. all variations with fixed endpoints, i.e.

$$\frac{d}{ds}\Big|_{s=0} I(X+sY, X+sY) = 0$$

for all vector fields Y along γ with $Y(a) = Y(b) = 0$.

Proof: We compute

$$\frac{d}{ds}\Big|_{s=0} I(X+sY, X+sY) = 2\, I(X,Y).$$

Then Prop 7. follows directly from Prop. 6. □

Remark: The Jacobi equation is the Euler-Lagrange equation for

$$I(X) = I(X,X).$$

In fact, one can consider the second variation for each critical point of a variational problem. The second variation then is a quadratic integral in the variation fields, and the second variation may be considered as a new variational problem. This new variational problem