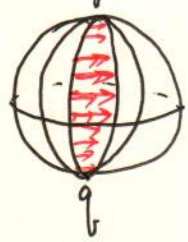
is called <u>accessary variational problem</u> of the original one.

Lecture 18 2017.04.25

§5. Conjugate Points and Minimizing Geodesics.

From Proposition 16 (p. 191) of §4, we see that if there exists nonzero Jacobi field U along the geodesic $\gamma : [a,b] \to M$ with $U(a) = U(b) = 0$, then ($I(U,U)=0$, i.e. I is not positive definite, and hence) $\gamma|_{[a,b]}$ may not be strictly local minimizing. This phenomena can ~~also been~~ be observed explicitly in $\mathbb{S}^2$.



For Any semicircle from the north pole p to the south pole q, $\exists$ nonzero Jacobi field U along it with $U(a) = U(b) = 0$.

each semicircle has the same length π.

<u>Definition</u> (Conjugate points) Let $\gamma : [a,b] \to M$ be a geodesic. For $t_0, t_1 \in [a,b]$, if there exists a Jacobi field $U(t)$ along γ that does not vanish identically, but satisfies

$$U(t_0) = U(t_1) = 0,$$

then t_0, t_1 are called <u>conjugate values</u> along γ. The <u>multiplicity</u> of t_0 and t_1 as conjugate values is defined as the dimension of the vector space consisting of all such Jacobi fields. We also say $\gamma(t_0), \gamma(t_1)$ are <u>conjugate points</u> of γ. (This terminology is ambiguous when γ has self-intersections). $\square$

Recall a Jacobi field U is determined by its initial values

$$U(t_0), \quad \dot{U}(t_0)$$

at any point t_0, Hence, the multiplicity of two conjugate values t_0, t_1 is clearly $\leq n$. Actually, it is $\leq n-1$. This is because a Jacobi field which is tangent to γ and vanish at t_0 will not vanish at t_1.

Proposition 8: Let $\gamma : [a,b] \to M$ be a geodesic with velocity field $T(t) = \dot{\gamma}(t)$.

(1) The vector field fT along γ is a Jacobi field if and only if f is linear

(2). Every Jacobi field U along γ can be written uniquely as

$$fT + U^{\perp},$$

where f is linear and $U^{\perp}$ is a Jacobi field perpendicular to γ.

(3) If a Jacobi field U along γ is perpendicular to γ at two points t_0 and t_1, then U is perpendicular to γ everywhere. In particular if $U(t_0) = U(t_1) = 0$, then U is perpendicular to γ everywhere.

Proof: (1) fT is Jacobi field $\Rightarrow$

$$0 = \nabla_T \nabla_T (fT) + \underbrace{R(fT, T)T}_{f R(T,T)T \, = 0} = f''(t) T$$

Hence f is linear.

(Note, if $f(t_0) = 0$, then $f(t_1) \neq 0$, $\forall t_1 \neq t_0$. if f is not identically 0)

(2) Let U be a Jacobi field along γ. we can write

$U = fT + U^{\perp}$ for some f and some $U^{\perp}$ with $\langle U^{\perp}, T \rangle = 0$.

U is Jacobi $\Rightarrow$

$$0 = \nabla_T \nabla_T (fT + U^{\perp}) + R(fT + U^{\perp}, T)T$$
$$= f'' T + \nabla_T \nabla_T U^{\perp} + R(U^{\perp}, T)T.$$

In particular, we have

$$0 = f'' + \langle \nabla_T \nabla_T U^{\perp}, T \rangle + \underbrace{\langle R(U^{\perp}, T)T, T \rangle}_{=0 \text{ by symmetry}}$$

~~$\langle \nabla_T \nabla_T U^{\perp}, T \rangle$~~

$$0 = \langle U^{\perp}, T \rangle \Longrightarrow 0 = \frac{d}{dt} \langle U^{\perp}, T \rangle = \langle \nabla_T U^{\perp}, T \rangle$$
$$\Rightarrow 0 = \frac{d}{dt} \langle \nabla_T U^{\perp}, T \rangle = \langle \nabla_T \nabla_T U^{\perp}, T \rangle$$

Hence $0 = f''$. (i.e. f is linear)

and $\nabla_T \nabla_T U^{\perp} + R(U^{\perp}, T)T = 0$, i.e. $U^{\perp}$ is a Jacobi field.

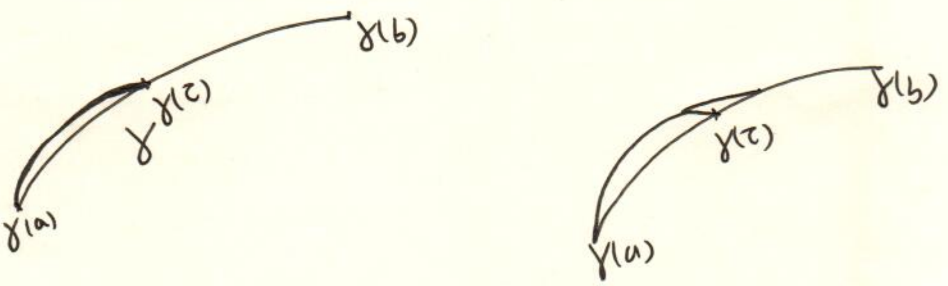
Uniqueness is obvious

(3) Write $U = fT + U^\perp$. Then $\langle U(t_0), T\rangle = \langle U(t_1), T\rangle = 0$ implies $f(t_0) = f(t_1) = 0$. Recall f is linear, we have $f \equiv 0$. Therefore $U = U^\perp$. □

Proposition 8 (3) shows that for the purpose of investigating conjugate values, we need consider only normal Jacobi fields.

Conjugate points play an important role in the study of local minima for length. A geodesic $\gamma : [a,b] \to M$ can not locally minimize length if $\exists\ \tau \in (a,b)$ conjugate to a.

Intuitive argument:



$\gamma(\tau)$ conjugate to $\gamma(a)$, $\Rightarrow \exists$ a geodesic η from $\gamma(a)$ to $\gamma(\tau)$ with nearly the same length as $\gamma|_{[a,\tau]}$.

Then η followed by $\gamma|_{[\tau,b]}$ has nearly the same length as γ.

By the first curve has a corner, and can be shorten by replacing the corner with a minimal geodesic. Therefore γ is not a ~~ext~~ minimizing curve.

In fact we have the following theorem of Jacobi.

Theorem 4 (Jacobi) Let $\gamma : [a,b] \to M$ be a geodesic from $p = \gamma(a)$ to $q = \gamma(b)$.

(1) If there is no conjugate points of p along γ, then there exists $\varepsilon > 0$ so that for any piecewise smooth curve $\Phi_c : [a,b] \to M$

from p to q satisfying $d(\gamma(t), c(t)) < \varepsilon$, we have

$$L(c) \geq L(\gamma),$$

with equality holds if and only if c is a reparametrization of γ.

(2) If there exists $\bar{t} \in (a,b)$ so that $\bar{q} = \gamma(\bar{t})$ is a conjugate point of p, then there is a proper variation of γ so that

$$L(\gamma_s) < L(\gamma)$$

for any $0 < |s| < \varepsilon$.

The above results are direct consequences of the corresponding properties of index forms, which will be discussed in the next subsection.

Next, we derive a characterization of the conjugate points in terms of critical point of the exponential map.

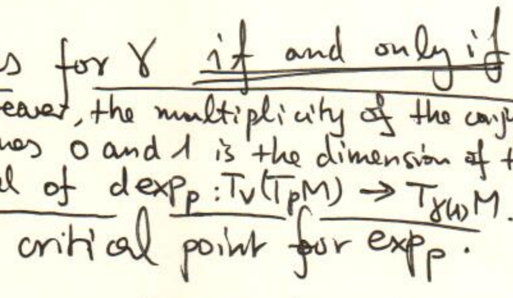
Theorem 5. Let $\gamma : [0,1] \to M$ be a geodesic with $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = V \in T_pM$, so that γ can be described as

$$t \longmapsto \exp_p tV.$$

Then 0 and 1 are conjugate values for γ if and only if V is a critical point of $\exp_p$. Moreover, the multiplicity of the conjugate values 0 and 1 is the dimension of the kernel of $d\exp_p : T_V(T_pM) \to T_{\gamma(1)}M$.

Proof: "$\Leftarrow$" Suppose that $V \in T_pM$ is a critical point for $\exp_p$. That is $0 = d\exp_p(V)(X)$ for some nonzero $X \in T_V(T_pM)$.

Let c be a path in T_pM with $c(0) = V$, $\dot{c}(0) = X$.



We put

$$F(t,s) = \exp_p t(c(s)), \quad t \in [0,1].$$

Then $F(t,0) = \exp_p tV$ ~~is a geo~~ $= \gamma$, and $\gamma_s(t) = \exp_p tc(s)$ is a geodesic

That is, F is a geodesic variation of γ. So the variational field

$$U(t) := \frac{\partial}{\partial s}\Big|_{s=0} \exp_p tc(s)$$

is a Jacobi field along γ. We compute $U(0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p 0 = 0$,

and $U(1) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p c(s) = d\exp_{p\,(c(0))}(\underset{X}{\underset{\shortparallel}{\dot c(0)}}) = d\exp_{p\,(v)}(X) = 0$

Next, we hope to show U is not identically zero. This is because

$$\dot U(0) = \widetilde{\nabla}_{\frac{\partial}{\partial t}} U(t)\Big|_{t=0} = \widetilde{\nabla}_{\frac{\partial}{\partial t}}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \exp_p tc(s) = \widetilde{\nabla}_{\frac{\partial}{\partial s}}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \exp_p tc(s)$$

$$= \widetilde{\nabla}_{\frac{\partial}{\partial s}}\Big|_{s=0} c(s)$$

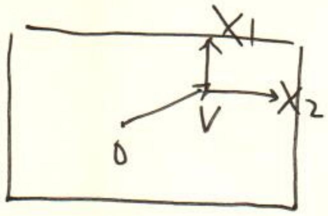
(the covariant derivative of the vector field $s \mapsto c(s)$ along the const. curve $s \mapsto p$).

$$= \dot c(0) = X \neq 0.$$

Therefore, we show 0 and 1 are conjugate values for γ.

("$\Rightarrow$") We argue by contridiction. Suppose V is not a critical point for $\exp_p$. If $X_1, \dots, X_n \in T_V(T_pM)$ are n linealy indep. vectors, then $d\exp_{p\,(V)}(X_1), \dots, d\exp_{p\,(V)}(X_n) \in T_{\gamma(1)}M$ are also linearly indep.

Choose paths $c_1, \dots, c_n$ in T_pM with



$$\begin{cases} c_i(0) = V \\ \dot c_i(0) = X_i, \quad i = 1, \dots, n. \end{cases}$$

And $F(t,s) := \exp_p tc_i(s)$

are geodesic variations of γ with variational fields $V_i(t)$.

The V_i are Jacobi fields along γ which vanish at 0.

Moreover, the $V_i(1) := d\exp_{p\,(V)}(X_i)$ are n indep., so no nontrivial linear combination of the V_i can vanish at 1.

Since the vector space of Jacobi fields along γ which vanish at 0 has dimension exactly n, it follows that no non-zero Jacobi field along γ vanishes at 0 and also at 1. $\square$

§6. Index forms.

In this section, we discuss the minimizing property of a geodesic via Index forms. For that purpose, we need consider a piecewise C^∞ proper variation of a geodesic γ. That is, we compare the length of a geodesic $\gamma : [a,b] \to M$ with the length of any piecewise C^∞ curve from $\gamma(a)$ to $\gamma(b)$. The corresponding variational field of γ is then a piecewise C^∞ vector field along γ.

Recall our calculations for the second variation formula (SVF), the result is the same as the case of smooth variation:

$$\frac{\partial^2}{\partial v \partial w}\Big|_{(v,w)=(0,0)} E(v,w) = \langle \nabla_W V, T\rangle\Big|_a^b + \int_a^b \langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle \, dt.$$

where $\langle \nabla_W V, T\rangle\big|_a^b = 0$ when the variation is proper.

Definition (Index form). The index form of a geodesic γ is

$$I(V,W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle\right) dt$$

where V, W are two piecewise smooth vector fields along γ.

Remark: (1) If V, W are C^∞ on each $[t_i, t_{i+1}]$ where

$$a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$$

is a subdivision of $[a,b]$. Then by integration by parts,

$$\begin{aligned} I(V,W) &= \int_a^b \langle \nabla_T V, \nabla_T W\rangle - \langle R(W,T)T, V\rangle \, dt \\ &= \sum_{i=0}^{k} \langle V, \nabla_T W\rangle\Big|_{t_i}^{t_{i+1}} + \int_a^b \langle -\nabla_T \nabla_T W, V\rangle - \langle R(W,T)T, V\rangle \, dt \end{aligned}$$

$\Rightarrow$

$$\boxed{\begin{aligned} I(V,W) = &-\int_a^b \langle \nabla_T \nabla_T W + R(W,T)T, V\rangle dt + \langle \nabla_T W, V\rangle\Big|_a^b \\ &- \sum_{j=1}^{k} \langle \nabla_{T(t_j^+)} W - \nabla_{T(t_j^-)} W, V\rangle \qquad (***) \end{aligned}}$$

(2). Note for a proper variation

$$\frac{\partial^2}{\partial v \partial w}\Big|_{(v,w)=(0,0)} E(v,w) = I(V,W).$$

Let $\mathcal{V}$:= the set of all piecewise smooth vector fields along $\gamma : [a,b] \to M$.

and $\mathcal{V}_0 := \{ X \in \mathcal{V} \mid X(a) = 0, X(b) = 0 \}$.

We need extend Proposition 6 (P.191) to piecewise smooth vector fields.

Proposition 6' Let $\gamma : [a,b] \to M$ be a geodesic and $U \in \mathcal{V}$. Then U is a Jacobi field if and only if $I(U,Y) = 0$, $\forall Y \in \mathcal{V}_0$.

Proof: Note that, comparing with Propo. 6. (p.191), we here have $U \in \mathcal{V}$ may be piecewise smooth, and so does Y. However, a Jacobi field is smooth. (The result and proof here is very much similar in spirit to the characterization of geodesic (see Exercise 6.2); A piecewise smooth curve c is a geodesic if and only if, for every proper variation F of c, we have $E'(0) = 0$.)

($\Rightarrow$) If u is a Jacobi field, then $I(\underset{\text{smooth}}{U}, \underset{\text{piecewise}}{Y}) = 0$, $\forall Y \in \mathcal{V}_0$

$$I(U,Y) = \int_a^b \left(\langle \nabla_T U, \nabla_T Y \rangle - \underbrace{\langle R(Y,T)T, U \rangle}_{= \langle R(U,T)T, Y \rangle} \right) dt$$

$$\overset{\text{Rmk (1)}}{\underset{(***)}{=}} - \int_a^b \langle \nabla_T \nabla_T U + R(U,T)T, Y \rangle + \underbrace{\langle \nabla_T U, Y \rangle \Big|_a^b}_{\substack{=0 \\ \text{since } Y \in \mathcal{V}. \\ Y(a)=0=Y(b)}}$$

$$+ \sum_{j=1}^{k} \langle \underbrace{\nabla_{T(t_j^+)} U - \nabla_{T(t_j^-)} U}_{=0 \text{ since } U \text{ smooth}}, Y \rangle$$

$$\overset{U \text{ is Jacobi}}{=} 0.$$

$a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$

($\Leftarrow$) Assume $I(U,Y) = 0$, $\forall Y \in \mathcal{V}_0$. Let $f : [a,b] \to \mathbb{R}$ be a smooth function s.t. $f(t_i) = 0$, $i = 0, \ldots, k+1$.

and $f > 0$ otherwise. Set $W = U$, $V = Y = f(\nabla_T \nabla_T U + R(U,T)T)$.

Note that Y is well-defined and $Y \in \mathcal{V}_0$.

Therefore

$$0 = I(U,Y) \overset{(***)}{=} -\sum_i \int_{t_i}^{t_{i+1}} f(t)\, \left|\nabla_T\nabla_T U + R(U,T)T\right|^2 dt$$

Hence, we have $\nabla_T\nabla_T U + R(U,T)T = 0$ on each $[t_i, t_{i+1}]$.

That is, "piecewisely", U is a Jacobi field. $(\star)$

Next, for any $j = 1, \dots, k$, let $Y \in \mathcal{V}_0$ s.t.

$$\begin{cases} Y(t_i) = 0, & \forall\, i \neq j \\ Y(t_j) = \nabla_{T(t_j^+)} U - \nabla_{T(t_j^-)} U. \end{cases}$$

Then $0 = I(U,Y) = \sum_j \left|\nabla_{T(t_j^+)} U - \nabla_{T(t_j^-)} U\right|^2$

Hence $\nabla_{T(t_j^+)} U = \nabla_{T(t_j^-)} U$.

Therefore U is a C^1 vector field along γ. ~~Moreover~~ Combining with the fact $(\star)$ and using the uniqueness of Jacobi fields with given initial data, we conclude U is the Jacobi field on $[a,b]$. □

<u>Remark</u>: ~~$I(V,W)$ is a bilinear symmetric form on the vector space $\mathcal{V}_0$~~

~~Since~~ Recall our previous discussions about SVF, we say the property "γ is locally minimizing" is equivalent to "$I(V,V) > 0$, $\forall\, 0 \neq V \in \mathcal{V}_0$". Since $I(V,W)$ is a bilinear, symmetric form on the vector space $\mathcal{V}_0$, the later condition is equivalent to say "I is positive definite on $\mathcal{V}_0$".

To illustrate the idea, we can compare the index form with the Hessian of a function $f: \mathbb{R}^n \to \mathbb{R}$. Consider a curve ξ in $\mathbb{R}^n$ with $\xi(0) \in \mathbb{R}^n$. Then the second order derivative of f along ξ is

$$\frac{d^2}{ds^2} f(\xi(s)).$$

Hessian of f valued at the vector $\dot\xi(0)$ is

$$\left.\frac{d^2}{ds^2} f(\xi(s))\right|_{s=0} = \mathrm{Hess} f\,(\dot\xi(0), \dot\xi(0))$$

In particular $\frac{d^2}{ds^2} f(\xi(s))\big|_{s=0}$ only depends on $\xi'(0)$

Once we know $\frac{d^2}{ds^2} f(\xi(s))\big|_{s=0}$, $\forall \xi$, then we have $\mathrm{Hess} f(v,v)$ for any v, and hence

$$\mathrm{Hess} f(v,w) = \frac{1}{2}\left(\mathrm{Hess} f(v+w, v+w) - \mathrm{Hess} f(v,v) - \mathrm{Hess} f(w,w)\right).$$

Analogously, we replace $\mathbb{R}^n$ by the space $\mathcal{C}$ of <u>all curves $\gamma:[a,b]\to M$</u>

Given a "point" of $\mathcal{C}$, i.e., a curve $\gamma \in \mathcal{C}$, consider a "curve" through it, i.e. a 1-parameter family of curves $\{\gamma_s\}$. Let E be a function

$$E : \mathcal{C} \to \mathbb{R}.$$

The restriction $E \circ \gamma_s := E(s)$, and

$$\frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0} = \frac{d^2}{ds^2} E(s)\Big|_{s=0} = \text{"}\mathrm{Hess}\, E^{*}(V(t), V(t))\text{"}$$

By polarization, one have "$\mathrm{Hess}\, E^{*}(V,W)$", the Hessian of E on the "Hilbert space of curves". All ~~formly~~ formal discussion here can be made rigorous.

In particular, when considering $\mathcal{C}_0$ of all curves $c:[a,b]\to M$ s.t. $c(a)=\gamma(a)$, $c(b)=\gamma(b)$, the "Hessian of E" is given by the <u>index form</u>.

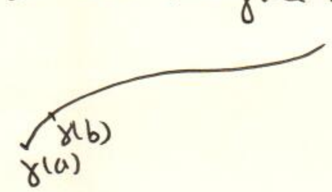
Next, our aim is discuss the relation between

$\boxed{\text{Algebraic propertyies of the index form}}$ of the geodesic γ.

and

$\boxed{\text{Minimizing properties of the geodesic } \gamma}$

Given a normal geodesic $\gamma:[a,b]\to M$, we can imagine the end point $\gamma(b)$ move from $\gamma(a)$ slowly to $\gamma(\infty)$. When $|b-a|$ is small enough,



$\gamma|_{[a,b]}$ is minimizing. Hence we can expect I is positive definite on $\mathcal{V}_0$.

By the rough idea we explained before Theorem 4 (p. 195), when $|b-a|$ is large, s.t. there is a conjugate value of a in (a,b), $\gamma|_{[a,b]}$ is not (locally) minimizing, then we can expect $\exists X$ s.t. $I(X,X)<0$.

In the case of a and b are conjugate values of γ, we have from Prop. 6', for any Jacobi field U along γ with $U(a)=U(b)=0$, we have

$$I(U,U)=0.$$

<u>Theorem 6</u>. Let $\gamma:[a,b]\to M$ be a geodesic from $p=\gamma(a)$ to $q=\gamma(b)$.

(1) $p=\gamma(a)$ has no conjugate point along γ $\Leftrightarrow$ the index form I is positive definite on $\mathcal{V}_0$.

(2) $q=\gamma(b)$ is a conjugate point of p along γ, and $\forall\, t\in(a,b)$, $\gamma(a)$ and $\gamma(t)$ are not conjugate point. (i.e. q is the first conjugate point of p).
$\Leftrightarrow$ I is positive semidefinite but not positive definite on $\mathcal{V}_0$.

(3) $\exists\, \bar{t}\in(a,b)$, s.t. $p=\gamma(a)$ and $\bar{q}=\gamma(\bar{t})$ are conjugate points
$\Leftrightarrow$ $I(X,X)<0$ for some $X\in\mathcal{V}_0$.

<u>Remark</u>. Theorem 6 tells if if $\gamma(a)$ has no conjugate point along $\gamma|_{[a,b]}$, then for any $[\alpha,\beta]\subset[a,b]$, $(\alpha<\beta)$, $\gamma(\alpha)$ also has no conjugate point along $\gamma|_{[\alpha,\beta]}$. Since otherwise, let $\tilde{J}$ be a nonzero Jacobi field along $\gamma|_{[\alpha,\beta]}$ with $\tilde{J}(\alpha)=0=\tilde{J}(\beta)$. Let I_r^s be the index form of $\gamma|_{[r,s]}$. Then let $J|_{[a,\alpha]}\equiv 0\equiv J|_{[\beta,b]}$, $J|_{[\alpha,\beta]}=\tilde{J}$

$$I_a^b(J,J)=I_a^\alpha(0,0)+I_\alpha^\beta(\tilde{J},\tilde{J})+I_\beta^b(0,0)$$
$$=I_\alpha^\beta(\tilde{J},\tilde{J})=0.$$

Hence (1) tells, $p=\gamma(a)$ does have a conjugate along $\gamma|_{[a,b]}$ $\lightning$. $\square$

Lecture 19 Thm 6(2) proved 2017.04.27

To show Thm 6 (1), we first prove the following useful Lemma.

<u>Lemma 1</u>: Let $\gamma:[a,b]\to M^n$ be a geodesic, and $\gamma(a)$ has no conjugate point along γ. Then for any $V_a\in T_{\gamma(a)}M$ and

$V_b \in T_{\gamma(b)}M$, there exists a unique Jacobi field U such that

$$U(a) = V_a, \quad U(b) = V_b .$$

Proof: By Prop 4 (2). (p.188), the ~~vector~~ vector space of all Jacobi fields along γ is of dimension $2n$. Let $\mathcal{J}'$ be the subspace of Jacobi fields U with $U(a) = V_a$. Then $\dim \mathcal{J}' = n$. Note that $T_{\gamma(b)}M$ is also a vector space with $\dim T_{\gamma(b)}M = n$. In fact, the linear transformation

$$A : \mathcal{J}' \longrightarrow T_{\gamma(b)}M$$
$$U \longmapsto U(b)$$

is injective. This is because if we have $U_1, U_2 \in \mathcal{J}'$ s.t.

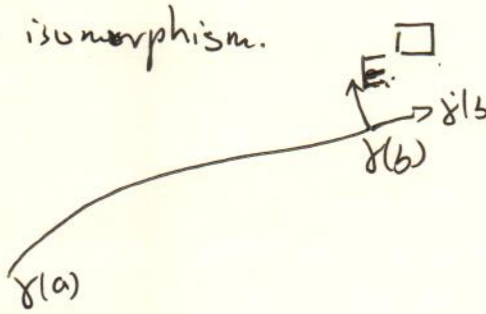
$$U_1(b) = U_2(b).$$

Then $U_1 - U_2$ is ~~also~~ again a Jacobi field along γ. We check

$$U_1 - U_2\,(a) = 0, \quad U_1 - U_2\,(b) = 0.$$

Since $\gamma(a)$ and $\gamma(b)$ are not conjugate points, we have $U_1 - U_2 \equiv 0$. therefore A is injective, and hence, an isomorphism. $\square$

Proof of Thm 6 (1) $(\Longrightarrow)$



let $\{\dot\gamma(b), E_2, \cdots, E_n\}$ be an orthonormal basis of $T_{\gamma(b)}M$. From Lemma 1, $\exists!$ Jacobi field J_i along γ s.t.

$$J_i(0) = 0, \quad J_i(b) = E_i, \quad i = 2, \cdots, n.$$

Moreover, Prop. 8 (3). (p.194) tells $\langle J_i(t), \dot\gamma(t)\rangle = 0, \ \forall t \in [a,b]$

By the argument in the proof of Thm 5. (p.196), $\{J_i(t)\}$ are linearly independent in ~~at~~ any ~~point~~ $T_{\gamma(t)}M$.

Note $I(f\dot\gamma(t), f\dot\gamma(t)) = \int_a^b \langle \dot f(t)\dot\gamma(t), \dot f(t)\dot\gamma(t)\rangle dt \geq 0$, and "$=$" $\Leftrightarrow f \equiv 0$

For any $U \in \mathcal{V}_0$, we ~~can~~ only consider write $U = f^i J_i$ for some fcts f^i s.t. $f^i(a) = f^i(b) = 0$. Next, we compute

$$I(U,U) = \int_a^b \langle \nabla_T (f^i J_i), \nabla_T (f^j J_j)\rangle - \langle R(f^i J_i, T)T, f^j J_j\rangle \, dt$$

$$= \underbrace{\int_a^b \langle \dot f^i J_i, \dot f^j J_j\rangle dt}_{A} + \underbrace{\int_a^b \langle \dot f^i J_i, f^j \nabla_T J_j\rangle}_{B} + \underbrace{\int_a^b \langle f^i \nabla_T J_i, \dot f^j J_j\rangle}_{C}$$

$$+ \underbrace{\int_a^b f^i f^j \langle \nabla_T J_i, \nabla_T J_j\rangle dt}_{D} - \underbrace{\int_a^b f^i f^j \langle R(J_i,T)T, J_j\rangle dt}_{E}.$$

Observe that

$$D = \int_a^b f^i f^j \langle \nabla_T J_i, \nabla_T J_j\rangle dt$$

$$= \int_a^b \left\{ \frac{d}{dt}\left(f^i f^j \langle \nabla_T J_i, J_j\rangle \right) - \dot f^i f^j \langle \nabla_T J_i, J_j\rangle - f^i \dot f^j \langle \nabla_T J_i, J_j\rangle \right.$$

$$\left. - f^i f^j \langle \nabla_T \nabla_T J_i, J_j\rangle \right\} dt$$

$$= f^i f^j \langle \nabla_T J_i, J_j\rangle \Big|_a^b - \int_a^b f^i \dot f^j \langle \nabla_T J_i, J_j\rangle - C + E.$$

$$= - \int_a^b f^i \dot f^j \langle \nabla_T J_i, J_j\rangle - C + E$$

In fact, $\langle \nabla_T J_i, J_j\rangle = \langle J_i, \nabla_T J_j\rangle$. This is because

$$\langle \nabla_T J_i, J_j\rangle - \langle J_i, \nabla_T J_j\rangle \Big|_{t=0} = 0 \quad (\text{since } J_i(0) = J_j(0) = 0)$$

and $\frac{d}{dt}\left(\langle \nabla_T J_i, J_j\rangle - \langle J_i, \nabla_T J_j\rangle \right) = \langle \nabla_T\nabla_T J_i, J_j\rangle - \langle J_i, \nabla_T \nabla_T J_j\rangle$

$$= \langle R(J_i,T)T, J_j\rangle - \langle R(J_j,T)T, J_i\rangle = 0, \ \forall t.$$

Therefore $D = -B - C + E$, and hence

$$I(U,U) = \int_a^b \langle \dot f^i J_i, \dot f^j J_j\rangle dt \geq 0.$$

Moreover "=" holds $\Leftrightarrow$ $\left.\begin{array}{l}\dot f^i = 0 \\ f^i(0) = 0 = f^i(b)\end{array}\right\} \Leftrightarrow f^i \equiv 0. \Leftrightarrow U = 0.$

This proves the positive definiteness of I on $\mathcal{V}_0$. $\square$

<u>Proof of Thm 6 (2)</u> ($\Rightarrow$) Choose any $c \in (a,b)$. Pick a parallel orthornormal vector fields $\{\dot\gamma(t), E_2(t), \cdots, E_n(t)\}$.

Then any $U \in \mathcal{V}_0 = \mathcal{V}_0[a,b]$, since $I(f\dot\gamma, f\dot\gamma) \geq 0$, we only need consider.

$$U(t) = \sum_{i=2}^n f^i(t) E_i(t)$$

for some fcts f^i with $f^i(a) = f^i(b) = 0$.

Define $\tau: \mathcal{V}_0(a,b) \longrightarrow \mathcal{V}_0(a,c)$, by

$$\tau(V)(t) = \cancel{\sum_{i=1}^{n} f^i\left(\frac{b-a}{c-a}(t-a)\right)E_i}$$

$$= \sum_{i=1}^{n} f^i\left(a+\frac{b-a}{c-a}(t-a)\right) E_i\left(a+\frac{b-a}{c-a}(t-a)\right).$$

By Thm 6 (1) ($\Rightarrow$), we know $I^c(\tau(V), \tau(V)) > 0$.

We can check by definition that $\qquad = \int_a^b (\dot{f}^i)^2 - f^i f^j \langle R(E_i,T)T, E_j\rangle \, dt$

$$\lim_{c\to b} I^c(\tau(V), \tau(V)) = I(V,V) \geq 0.$$

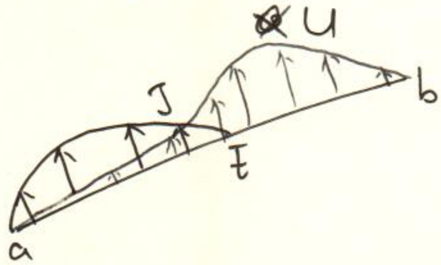
Hence I is positive semidefinite. We already explained that I is not positive definite, since for any nonzero Jacobi field U with $U(a) = U(b) = 0$, we have $I(U,U) = 0$. $\square$

<u>Proof of Thm 6 (3) ($\Rightarrow$)</u>. Let $\bar{t}$ is conjugate to a along γ, and there is a non-zero Jacobi field J along γ s.t. $J(a) = J(\bar{t}) = 0$

Let $\tilde{J}$ be the vector field along γ with

$\tilde{J}(t) = J(t)$, for $a \leq t \leq \bar{t}$

$\tilde{J}(t) = 0$, $\quad \bar{t} \leq t \leq b$.



Notice that the discontinuity of $\tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{J} = \nabla_T \tilde{J}$ since

$$\underbrace{\nabla_{T(\bar{t}^+)}\tilde{J}}_{=0} - \nabla_{T(\bar{t}^-)}\tilde{J} = -\nabla_{T(\bar{t}^-)}\tilde{J} \neq 0.$$

(since otherwise, together with $\tilde{J}(\bar{t}) = 0$, this implies $\tilde{J} \equiv 0$.)

Choose a vector field U along γ which satisfies

$$U(a) = 0 = U(b), \quad \langle U(\bar{t}), \nabla_{T(\bar{t}^+)}\tilde{J} - \nabla_{T(\bar{t}^-)}\tilde{J} \rangle = -1.$$

Define the vector field along γ

$$X := \frac{1}{c}\tilde{J} - cU. \quad \text{where } c \text{ is a small number.}$$

Then $\quad I(X,X) = \frac{1}{c^2} I(\tilde{J},\tilde{J}) - 2I(\tilde{J}, U) + c^2 I(U,U).$

where $\quad I(\tilde{J},\tilde{J}) = 0$ since $\tilde{J} \in \mathcal{V}_0(a,b)$.

$$I(\tilde{J}, U) \underset{(***)\,(p.198)}{=} -\langle U(\bar{t}), \nabla_{T(\bar{t}^+)}\tilde{J} - \nabla_{T(\bar{t}^-)}\tilde{J}\rangle = 1$$

Hence $I(X,X) = -2 + c^2 I(U,U)$

For sufficiently small c, this is <0. $\square$

The Thm 6 (1)($\Leftarrow$) follows from Thm 6 (2) $\Rightarrow$ & (3) $\Rightarrow$.

Similarly, Thm 6 (2) ($\Leftarrow$), Thm 6(3) $\Leftarrow$ are proved. $\square$

Let us mention a very useful lemma. Recall in Prop. 7 (p. 192), we have shown a Jacobi field U is the critical point of $I(X,X)$.

<u>Lemma 2</u>. (<u>Minimizing property</u> of Jacobi field) Let $\gamma : [a,b] \to M$ be a ~~normal~~ geodesic without conjugate points, U be a Jacobi field along γ, and X a piecewise C^∞ vector field along γ with

$$X(a) = U(a), \quad X(b) = U(b)$$

Then
$$I(U,U) \leq I(X,X)$$
where "=" holds iff $X = U$.

<u>Proof</u>: From (***) (p. 198), we see for any ~~W~~ piecewise C^∞ vector field W along γ, we have

$$I(U,W) = \langle \nabla_T U, W \rangle \Big|_a^b \qquad (\#)$$
(because U is smooth)

Since $X - U \in \mathcal{V}_0(a,b)$, Thm 6(1) ($\Rightarrow$) tells

$$0 \leq I(X-U, X-U) = I(X,X) + I(U,U) - 2I(X,U)$$
$$\overset{(\#)}{=} I(X,X) + \langle \nabla_T U, U \rangle \Big|_a^b - 2 \langle \nabla_T U, X \rangle \Big|_a^b$$
$U = X$ at a and b.
$$= I(X,X) - \langle \nabla_T U, U \rangle \Big|_a^b$$
$$= I(X,X) - I(U,U).$$

If $I(X,X) = I(U,U)$, we have $I(X-U, X-U) = 0$.

Theorem 6(1) tells $X - U = 0$. $\square$

<u>Remark</u>: In the above, we derive Lemma 2 from Thm 6(1) $\Rightarrow$. In fact, the converse is also true.

In fact, the results in Theorem 6 can be pushed forward much further to ~~be~~ the celebrated Morse Index Theorem.

We particularly observe that for a geodesic $\gamma : [a,b] \to M$

$\gamma(a)$ has a conjugate point in $(a,b) \Leftrightarrow \exists X \in \mathcal{V}_0(a,b), \ I(X,X) < 0$.

Definition (index and nullity of γ). We call for a geodesic $\gamma : [a,b] \to M$

$$\mathrm{ind}(\gamma) = \max\dim \left\{ \mathcal{A} \subset \mathcal{V}_0 \,\middle|\, \begin{array}{l} I \text{ is negatively definite on the} \\ \text{subspace } \mathcal{A} \end{array} \right\}$$

the index of γ. (i.e. $\forall X \in \mathcal{A}, I(X,X) < 0$).

We call

$$N(\gamma) = \dim \{ X \in \mathcal{V}_0 \mid I(X,Y) = 0 \ \forall Y \in \mathcal{V}_0 \}$$

the nullity of γ.

Remark: In fact $N(\gamma)$ is equal to the multiplicity of a and b as conjugate values. If $\gamma(b)$ is not conjugate to $\gamma(a)$, then $N(\gamma) = 0$. (By Prop. 6).

In this language, Thm 6(3) can be restated as

$$\boxed{\begin{array}{l} \exists \bar{t} \quad N(\gamma|_{[a,\bar{t}]}) \geq 1 \quad \Leftrightarrow \quad \mathrm{index}(\gamma) \geq 1. \\ \qquad\quad \Updownarrow \\ \text{number of conj. points of } \gamma(a) \geq 1. \end{array}}$$

A far-reaching generalization is the following celebrated Thm.

Morse index Theorem: The index of $\gamma : [a,b] \to M$ is the number of $\bar{t} \in (a,b)$ which are conjugate to a, each conjugate value being counted with its multiplicity. The index is always finite. That is

$$\mathrm{ind}(\gamma) = \sum_{a<t<b} N\left(\gamma|_{[a,t]}\right) < \infty.$$

In particular, $\gamma(a)$ has only finite many conjugate points along γ.

For the proof, one need to show the index of γ increases by at least v as t passes a conjugate value $\bar{t}$ with multiplicity v. This can be handled by essentially the same trick which was used in the proof of Thm 6(3). We refer to [WSY Chap 9] or [XLN, Chap. 12] for details of proof. (see [JJ, §4.3] for an analytic proof !!)

It is a good point to preflect our proof of Bonnet - Meyers Theorem (p. 156 - 159). We show that if $\sec \geq k > 0$, for a geodesic γ of length $l > \frac{\pi}{\sqrt{k}}$ we have for

$$V(t) = \sin\left(\frac{\pi}{l} t\right) E(t)$$

where $E(t)$ is a parallel vector field along γ,

$$I(V, V) < 0.$$

Note when $l = \frac{\pi}{\sqrt{k}}$, $\sec = k > 0$

$$V(t) = \sin(\sqrt{k}\, t)\, E(t)$$

is a Jacobi field along γ. (Exercise 11)

In particular, when $\sec = k > 0$, a geodesic γ of length $l > \frac{\pi}{\sqrt{k}}$ contains at least a conjugate point of $\gamma(0)$. Hence $\operatorname{ind}(\gamma) \geq 1$, and γ is not (locally) minimizing.

The proof of Bonnet - Myers tells when $\sec \geq k$, a geodesic of length $l > \frac{\pi}{\sqrt{k}}$ also contain at least one conjugate point, and $\operatorname{index}(\gamma) \geq 1$.

On the other hand, if $\sec = 0$ or $\sec = -k$, $k > 0$, the Jacobi field along γ with $J(0) = 0$ are linear combinations of $d\, t E(t)$ / $d \sinh(\sqrt{k}t) E(t)$ which will never vanish ~~somewhere~~ anywhere other than 0. Hence γ does not contain conjugate points. In fact, this is true for the case sec ≤ 0. This is the ~~topi~~ our next topic.

§7. Cartan-Hadamard Theorem

Recall in (IV) §5 (p.144. Cor 1) we have shown that when sec ≤ 0, every geodesic is locally minimizing. ~~In fact,~~ This indicates that no conjugate points exist in this setting.

Proposition 9. If all sectional curvatures of (M, g) are ≤ 0, then no two points of M are conjugate along any geodesic.

Proof: Let γ be a geodesic with velocity field $\overbrace{\ }^{T(t)=\dot\gamma(t)}$ along it. Let $U(t)$ be a Jacobi field along γ. Then

$$\nabla_T \nabla_T U + R(U,T)T = 0.$$

So $\langle \nabla_T \nabla_T U, U\rangle = -\langle R(U,T)T, U\rangle \ge 0$.

Therefore, $\frac{d}{dt}\langle \nabla_T U, U\rangle = \langle \nabla_T \nabla_T U, U\rangle + \langle \nabla_T U, \nabla_T U\rangle$

$$\ge 0,$$

~~However,~~ that is, $\langle \nabla_T U, U\rangle$ is non-decreasing.

~~Now if $U(t)$ vanishes at two different values, t_0 and t_1, then $\langle \nabla_T U, U\rangle$ vanishes at t_0 and t_1, so $\langle \nabla_T U, U\rangle$ must be 0 on the interval $[t_0, t_1]$. Clearly, this implies~~

~~$$\nabla_T U(t_0) = 0.$$~~

~~And, hence, $U \equiv 0$.~~ □

~~Remark: (Reflection of the proof). The key is to show~~

$$\frac{d}{dt}\langle \nabla_T U, U\rangle \ge 0$$

Notice that LHS $= \frac{1}{2}\frac{d^2}{dt^2}\langle U, U\rangle$.

That is, we have shown $\frac{d^2}{dt^2}|u(t)|^2 \geq 0$, i.e. $|u(t)|^2$ is convex

So $u(t_0) = 0 = u(t_1) \Rightarrow u \equiv 0$. $\square$

Remark: In fact, for a normal Jacobi field $u(t)$ with $u(0)=0$, define

$$f : (0,\infty) \to \mathbb{R} \quad \text{by} \quad f(t) = |u(t)| = \langle u(t), u(t)\rangle^{\frac{1}{2}}$$

At the values t with $u(t) \neq 0$, we compute

$$\dot{f}(t) = \frac{d}{dt} f(t) = \frac{\frac{d}{dt}\langle u(t), u(t)\rangle}{2\langle u(t), u(t)\rangle^{\frac{1}{2}}} = \frac{\langle \dot{u}(t), u(t)\rangle}{|u(t)|}.$$

$$\ddot{f}(t) = \frac{\left(\langle \ddot{u}(t), u(t)\rangle + \langle \dot{u}(t), \dot{u}(t)\rangle\right)|u(t)| - \langle \dot{u}(t), u(t)\rangle \frac{\langle \dot{u}(t), u(t)\rangle}{|u(t)|}}{|u(t)|^2}$$

$$= -\frac{\langle \dot{u}(t), u(t)\rangle^2}{|u(t)|^3} + \frac{|\dot{u}(t)|^2}{|u(t)|} - \frac{1}{|u(t)|}\langle R(u,T)T, u\rangle$$

(Cauchy-Schwarz $\langle \dot{u}(t), u(t)\rangle^2 \leq |\dot{u}(t)|^2 \cdot |u(t)|^2$)

$$\geq -\frac{|\dot{u}(t)|^2}{|u(t)|} + \frac{|\dot{u}(t)|^2}{|u(t)|} - \frac{1}{|u(t)|} K(u,T)\left(\langle u,u\rangle\langle T,T\rangle - \underbrace{\langle u,T\rangle^2}_{\substack{= \\ 0 \text{ since } u \\ \text{is normal}}}\right).$$

$$= -K(u,T)\underbrace{|u(t)|}_{= f(t)}.$$

That is, $\boxed{\frac{d^2}{dt^2} f(t) \geq -K(u,T) f(t).}$ $\quad f(t) := |u(t)|$.

$f(0) = 0$.

A comparison result:

$$\begin{cases} f''(t) \geq -\beta f(t) \\ f(0) = 0 \\ \dot{f}(0) = 1 \end{cases} \quad \text{and} \quad \begin{cases} g''(t) = -\beta g(t) \\ g(0) = 0 \\ \dot{g}(0) = 1. \end{cases}$$

Then $f(t) \geq g(t)$. $\left(\text{use } (f-g)'' \geq 0 \text{ and } (f-g)(0) = 0,\ \frac{d}{dt}(f-g)^{\circ}(0) = 0\right)$

~~Exercise~~ This is a very useful principle to ~~detect~~ investigate the geometry of a Rie-mfld via its Jacobi field and that of the space forms. Lecture 20. 2017.05.02 $\square$