conjugate points. In fact, this is true for the case sec $\le 0$. 

This is the ~~topi~~ our next topic.

## §7. Cartan-Hadamard Theorem

Recall in (IV) §5 (p.144. Cor 1) we have shown that when sec $\le 0$, every geodesic is locally minimizing. ~~In fact,~~ This indicates that no conjugate points exist in this setting.

Proposition 9. If all sectional curvatures of $(M, g)$ are $\le 0$, then no two points of $M$ are conjugate along any geodesic.

Proof: Let $\gamma$ be a geodesic with velocity field $T(t) = \dot\gamma(t)$ along it. Let $U(t)$ be a Jacobi field along $\gamma$. Then

$$\nabla_T \nabla_T U + R(U,T)T = 0.$$

So $\quad \langle \nabla_T \nabla_T U, U\rangle = -\langle R(U,T)T, U\rangle \ge 0.$

Therefore,

$$\frac{d}{dt}\langle \nabla_T U, U\rangle = \langle \nabla_T\nabla_T U, U\rangle + \langle \nabla_T U, \nabla_T U\rangle \ge 0,$$

~~However,~~ that is, $\langle \nabla_T U, U\rangle$ is non-decreasing.

Now if $U(t)$ vanishes at two different values, $t_0$ and $t_1$, then $\langle \nabla_T U, U\rangle$ vanishes at $t_0$ and $t_1$, so $\langle \nabla_T U, U\rangle$ must be $0$ on the interval $[t_0, t_1]$. Clearly, this implies $\nabla_T U(t_0) = 0$.

And, hence, $U \equiv 0$. $\square$

Remark. (Reflection of the proof). The key is to show

$$\frac{d}{dt}\langle \nabla_T U, U\rangle \ge 0$$

Notice that $\text{LHS} = \frac{1}{2}\frac{d^2}{dt^2}\langle U, U\rangle$.

That is, we have shown $\frac{d^2}{dt^2}|u(t)|^2 \geq 0$, i.e. $|u(t)|^2$ is convex

So $u(t_0) = 0 = u(t_1) \Rightarrow u \equiv 0$. $\square$

**Remark**: In fact, for a normal Jacobi field $u(t)$ with $u(0)=0$, define

$$f : (0,\infty) \to \mathbb{R} \quad \text{by} \quad f(t) = |u(t)| = \langle u(t), u(t)\rangle^{\frac{1}{2}}$$

At the values $t$ with $u(t) \neq 0$, we compute

$$\dot{f}(t) = \frac{d}{dt} f(t) = \frac{\frac{d}{dt}\langle u(t), u(t)\rangle}{2\langle u(t), u(t)\rangle^{\frac{1}{2}}} = \frac{\langle \dot{u}(t), u(t)\rangle}{|u(t)|}.$$

$$\ddot{f}(t) = \frac{\left(\langle \ddot{u}(t), u(t)\rangle + \langle \dot{u}(t), \dot{u}(t)\rangle\right)|u(t)| - \langle \dot{u}(t), u(t)\rangle \frac{\langle \dot{u}(t), u(t)\rangle}{|u(t)|}}{|u(t)|^2}$$

$$= -\frac{\langle \dot{u}(t), u(t)\rangle^2}{|u(t)|^3} + \frac{|\dot{u}(t)|^2}{|u(t)|} - \frac{1}{|u(t)|}\langle R(u,T)T, u\rangle$$

(Cauchy-Schwarz $\langle \dot{u}(t), u(t)\rangle^2 \leq |\dot{u}(t)|^2 \cdot |u(t)|^2$)

$$\geq -\frac{|\dot{u}(t)|^2}{|u(t)|} + \frac{|\dot{u}(t)|^2}{|u(t)|} - \frac{1}{|u(t)|} K(u,T)\left(\langle u,u\rangle\langle T,T\rangle - \underbrace{\langle u,T\rangle^2}_{\substack{= \\ 0 \text{ since } u \\ \text{is normal}}}\right).$$

$$= -K(u,T)\underbrace{|u(t)|}_{= f(t)}.$$

That is, $\boxed{\frac{d^2}{dt^2} f(t) \geq -K(u,T) f(t).}$ $\quad f(t) := |u(t)|$.

$f(0) = 0$.

A comparison result:

$$\begin{cases} f''(t) \geq -\beta f(t) \\ f(0) = 0 \\ \dot{f}(0) = 1 \end{cases} \quad \text{and} \quad \begin{cases} g''(t) = -\beta g(t) \\ g(0) = 0 \\ \dot{g}(0) = 1. \end{cases}$$

Then $f(t) \geq g(t)$. (use $(f-g)'' \geq 0$ and $(f-g)(0) = 0$, $\frac{d}{dt}(f-g)(0) = 0$).

<u>Exercise</u> This is a very useful principle to ~~detect~~ investigate the geometry of a Rie-mfld via its Jacobi field and that of the space forms. $\square$

Lecture 20. 2017.05.02

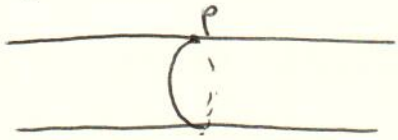
Theorem 7 (Cartan-Hadamard) A complete, simply-connected, $n$-dimensional Riemannian manifold $(M, g)$ with all sectional curvatures $\leq 0$ is diffeomorphic to $\mathbb{R}^n$; more precisely,

$$\exp_p : T_pM \to M$$

is a diffeomorphism.

Remark. In 1898, Hadamard proved such properties for a complete, simply-connected surface with non-positive Gauss curvature. In 1928, E. Cartan extended it to $n$-dim'l Riemannian manifolds. In fact, Hadamard's result has been proved by von Mongoldt in 1881.

The assumption of "simply-connectivity" is necessary. For example, the cylinder $C \equiv \{(x,y,z) \in \mathbb{R}^3 : x^2+z^2=1, y \in \mathbb{R}\}$




is complete and with sectional curvature zero. Its exponential map $\exp_p : T_pC \to C$ is a non-trivial covering map.

An important feature of Thm 7 is that it not only asserts that $M$ and $\mathbb{R}^n$ are diffeomorphic, but also give the diffeomorphism map explicitly. Recall we have mention Gromoll-Meyer (1969) Theorem ~~proved that~~ any non-compact complete Rie. mfld $(M^n, g)$ with positive sectional curvature is diffeomorphic to $\mathbb{R}^n$. But in this case, the diffeomorphism map is not necessarily given ~~by~~ explicitly by $\exp_p$. This is a big difference between our understanding about nonpositively curved simply-connected complete Rie mfld and positively curved non-compact complete Rie mfld, although their topology are both trivial.

Thm 7 is a direct consequence of Proposition 9 and the following general result.

<u>Theorem 8</u>: Let $(M, g)$ be a complete, connected, $n$-dim'l Riemannian manifold, and let $p$ be a point of $M$ such that <u>no point of $M$ is conjugate to $p$</u> along any geodesic. Then

$$\exp_p : T_pM \to M$$

is a <u>covering map</u>. In particular, if $M$ is simply-connected, then $M$ is diffeomorphic to $\mathbb{R}^n$.

<u>Proof</u>: We first make it clear what the assumption "$p$ has no conjugate point" tells us.

$$\exp_p : T_pM \to M$$

metric $g$

Then we have the tensor $(\exp_p)^* g$ on $T_pM$ which is defined as

$\forall V, W \in T_x(T_pM)$, $\forall x \in T_pM$.

$$(\exp_p)^* g(V, W) := g\big((d\exp_p)_{(x)}(V), (d\exp_p)_{(x)}(W)\big)$$

"$p$ has no conjugate point" $\Rightarrow$ $(d\exp_p)_{(x)} : T_x(T_pM) \to T_{\exp_p x}M$ is 1-1.

Therefore $(\exp_p)^* g(V, V) = 0 \Leftrightarrow (d\exp_p)_x(V) = 0 \Leftrightarrow V = 0$.

and, hence, $(\exp_p)^* g$ is a Riemannian metric on $T_pM$.

That is,
$$\exp_p : (T_pM, (\exp_p)^* g) \longrightarrow (M, g)$$
is a local isometry.

Moreover, we claim $(T_pM, (\exp_p)^* g)$ is complete.

This is because, all straight lines through $0 \in T_pM$ are geodesics of $(T_pM, (\exp_p)^*g)$ since their images under the local isometry $\exp_p : T_pM \to M$ are geodesics in $M$. That is, all geodesics through $0 \in T_pM$ can be defined for all $t$. It follows that $T_pM$ is geodesic complete and, hence, complete, by Hopf-Rinow.

In conclusion, "<u>$p$ has no conjugate point</u>" $\Rightarrow$

" $\exp_p : T_pM \to M$ is a local isometry and $T_pM$ is complete".

Then Theorem 8 is a consequence of the following lemma:

<u>Lemma 3</u>. Let $M$ and $N$ be connected Rie. mflds with $M$ complete, and let $\phi : M \to N$ be a <u>local isometry</u>. Then $N$ is <u>complete</u> ① and $\phi$ is <u>a covering map</u> ② <u>onto</u> ③ $N$

<u>Remark</u>: Lemma 3 has 3 (① - ③) conclusions. Note that the completeness of $M$ is needed. For example the inclusion map $i : B(0,1) \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a local isometry but the open disk $B(0,1)$ is not complete. $i$ is not a covering map.

<u>Proof of Lemma 3</u>:

① <u>N is complete</u>: We will show any geodesic on $N$ is defined for all $t$. ~~Let $p_0 \in M$.~~ For any geodesic $\gamma : (-\varepsilon, \varepsilon) \to N$. ~~s.t. $\gamma(0) = \phi(p_0)$~~ there exists a $p_0 \in M$ s.t.

$$\gamma(0) = \phi(p_0)$$

Then we find <u>the</u> geodesic $c$ in $M$ with

$$c(0) = p_0, \qquad \dot{c}(0) = (d\phi_{p_0})^{-1} \dot{\gamma}(0)$$

This is possible since $\phi$ is a local isometry.

On can check the curve $\phi \circ c$ is a geodesic (since $\phi$ is a local isometry which preserves geodesic), and

$$\phi \circ c(0) = \phi(p_0), \quad (\phi \dot{\circ} c)(0) = d\phi_{p_0}\, \dot{c}(0) = \dot{\gamma}(0).$$

Hence $\gamma = \phi \circ c$.

M is complete $\Rightarrow$ c is defined for all t $\Rightarrow$ $\gamma$ is defined for all t.

Hopf-Rinow tells N is complete.

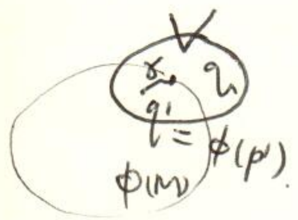
② $\phi$ is <u>onto</u> N : That is, we have to show $\phi(M) = N$.

Since $\phi$ is everywhere regular (i.e. $\forall p \in M$, $d\phi_p$ ~~is~~ one-one), by inverse function theorem, $\phi$ is open, and, in particular, $\phi(M)$ is open.

In fact, <u>$\phi(M)$ is also closed</u>, and, hence, $\phi(M) = N$.

$\Downarrow$ reasoning:

Let $q \in \overline{\phi(M)}$, and let V be a totally normal neighborhood of $q$. There is a $q' \in V$ of the form $q' = \phi(p')$ for $p' \in M$.




let $\gamma$ be <u>the</u> geodesic in V s.t.

$$\gamma(0) = q', \quad \gamma(1) = q.$$

Consider <u>the</u> geodesic c in M s.t. $c(0) = p'$, $\dot{c}(0) = (d\phi_{p'})^{-1}\dot{\gamma}(0)$

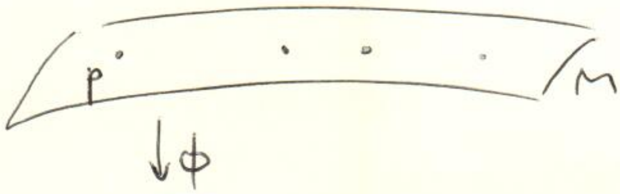
Then $\gamma = \phi \circ c$.

Define $p = c(1)$, then

$$\phi(p) = \phi(c(1)) = \phi \circ c(1) = \gamma(1) = q.$$

Therefore $q \in \phi(M)$. That is $\overline{\phi(M)} \subset \phi(M)$, so $\phi(M)$ is closed.

③ <u>$\phi$ is a covering map</u>:




For fixed $q \in N$, let

$$B(0, 2\varepsilon) := \{Y \in T_qN : \|Y\| < 2\varepsilon\} \subset T_qN.$$

~~be a nei~~ where $\varepsilon>0$ is small enough such that $\exp_q$ is a diffeomorphism. Suppose $p \in \phi^{-1}(q)$. ~~Then~~ We have

$$\begin{array}{ccc} B(0,2\varepsilon)\subset T_pM & \xrightarrow{\ d\phi_p\ } & B(0,2\varepsilon)\subset T_qN \\ \Big\downarrow \exp_p & & \Big\downarrow \exp_q \\ B_p(2\varepsilon)\subset M & \xrightarrow{\ \phi\ } & B_q(2\varepsilon)\subset N \end{array} \quad (\star)$$

diffeomorphism.

(it is well defined since M is complete.)

where $\phi \circ \exp_p = \exp_q \circ d\phi_p$.

This is because, $\forall\, V \in B(0,1) \subset T_pM$,

$\gamma(t) = \phi \circ \exp_p(tV)$, $t \in [0, 2\varepsilon)$ is a geodesic in $N$

s.t. $\gamma(0) = \phi(p) = q$

$\dot\gamma(0) = d\phi_p(V)$

On the other hand,

$c(t) = \exp_q \circ d\phi_p(tV) = \exp_q\left(t\, d\phi_p(V)\right)$ is a geodesic in $N$ with

$c(0) = \exp_q 0 = q$

$\dot c(0) = d\phi_p(V)$.

Hence $\exp_q \circ d\phi_p(tV) = \phi \circ \exp_p(tV)$, $\forall\, V \in B(0,1) \subset T_pM$, $\forall\, t \in [0, 2\varepsilon)$

$$\Rightarrow \exp_q \circ d\phi_p \big|_{B(0,2\varepsilon)\subset T_pM} = \phi \circ \exp_p \big|_{B(0,2\varepsilon)\subset T_pM}. \quad (\star\star\star)$$

That is, the diagram $(\star)$ commutes.

Therefore, we have (using the fact $\exp_q : B(0,2\varepsilon) \subset T_qN \to B_q(2\varepsilon) \subset N$ is a diffeomorphism)

~~$f = \exp_p \circ (d\phi_p)^{-1} \circ (\exp_q)^{-1}$ is a well defined map~~

~~s.t. $\phi \circ f = f \circ \phi = id$.~~

the LHS of $(\star\star\star)$ is a diffeomorphism. Since $\exp_p : B(0,2\varepsilon) \subset T_pM$

$\to B_p(2\varepsilon) \subset M$ is surjective, ~~***~~ and $\phi \circ \exp_p$ is a diffeomorphism

by (***), we have ~~exp_p is a diff~~

$$B(0,2\varepsilon) \subset T_pM \xrightarrow{\exp_p} B_p(2\varepsilon) \subset M$$

is a diffeomorphism.

Therefore, (***) $\Rightarrow$ $\phi = \exp_q \circ d\phi_p \circ \exp_p^{-1}$ is also an ~~iso~~ diffeomorphism.

~~Notice that~~,

Now let, $W := \exp_q(B(0,\varepsilon)) \subset N$

and $\forall p \in M$, $W_p := \exp_p(B(0,\varepsilon)) \subset M$

We claim that

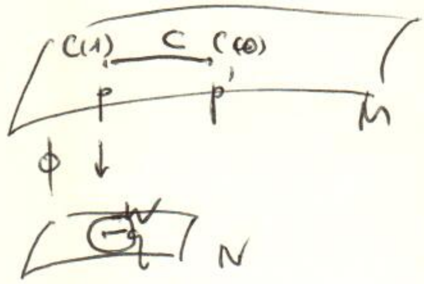
$$(1) \qquad \phi^{-1}(W) = \bigcup_{p \in \phi^{-1}(q)} W_p .$$

Note that $\phi : W_p \to W$ is a diffeomorphism by our previous argument. So in order to show $\phi : M \to N$ is a covering map, we only need to show the claim and

$$(2) \qquad W_{p_i} \cap W_{p_j} \neq \phi \quad \forall\, p_i, p_j \in \phi^{-1}(q), p_i \neq p_j .$$

Proof of (1): ~~$\phi(W_p) \to W$~~ ~~$\phi|W_p$~~ $\phi : W_p \to W$ is a diffeomorphism

tells $\bigcup_{p \in \phi^{-1}(q)} W_p \subset \phi^{-1}(W)$




Now $\forall\, p' \in \phi^{-1}(W)$,

let $\gamma$ be the geodesic in $W$ with

$\gamma(0) = \phi(p')$, $\gamma(1) = q$

and of length $d(\phi(p'), q)$.

Let $c$ be the geodesic in $M$ with $c(0) = p'$, $\dot c(0) = (d\phi_{p'})^{-1} \dot\gamma(0)$

Then $\gamma = \phi \circ c$. $M$ is complete $\Rightarrow$ $\underset{\overset{\shortparallel}{q}}{\gamma(1)} = \phi \circ c(1)$ is well defined.

In particular,

$$c(1) \in \phi^{-1}(q)$$

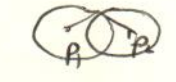
and $p' = c(0) \in W_{c(1)}$.

This implies $\phi^{-1}(W) \subset \bigcup_{p \in \phi^{-1}(q)} W_p$.

Hence, we prove the claim. (1)

Proof of (2): Suppose $\exists\ p_1, p_2 \in \phi^{-1}(q)$, $p_1 \neq p_2$ s.t. $W_{p_1} \cap W_{p_2} \neq \phi$,

then we have



$$p_2 \in \exp_{p_1}\big(\overbrace{B(0, 2\varepsilon)}^{\subset T_{p_1}M}\big)$$

But $\phi$ is a diffeomorphism on the $B(0,2\varepsilon) \subset T_{p_1}M \to B_{p_1}(2\varepsilon) \subset M$.

Hence $\phi(p_1) = \phi(p_2) \Rightarrow p_1 = p_2$. $\lightning$ $\square$

## §8. Uniqueness of simply-connected space forms.

Now we can prove the "uniqueness part" of Theorem 1 (p.169) which we stated in the very beginning of this Chapter. In fact, we can prove.

Theorem 9. (Uniqueness) Let $(M, g)$ and $(\bar M, \bar g)$ be two $n$-dim'l simply-connected space-form with sectional curvature $c \in \mathbb{R}$. Let $p \in M$, $\bar p \in \bar M$, $\{e_1, \dots, e_n\}$, $\{\bar e_1, \dots, \bar e_n\}$ be orthonormal basis of $T_pM$, $T_{\bar p}\bar M$, respectively. Then there exists a unique isometry $\varphi : M \to \bar M$ such that $\varphi(p) = \bar p$, $d\varphi_p(e_i) = e_i'$, $\forall i$.

Proof: Since $K(Ag) = \frac{1}{A} K(g)$, we only need consider the cases $c = 0, +1, -1$. We first show the existence of such an isometry.

Case 1. $c = 0$ or $-1$. By Cartan-Hadamard, the maps

$$\exp_p : T_pM \to M \quad \text{and} \quad \exp_{\bar p} : T_{\bar p}\bar M \to \bar M$$

are both diffeomorphisms

Let $\Phi$ be the unique isometry from $T_pM$ to $T_{\bar p}\bar M$ (as inner product) such that

$$\begin{tikzcd} T_pM \arrow[r, "\Phi"] \arrow[d, "\exp_p"] & T_{\bar p}\bar M \arrow[d, "\exp_{\bar p}"] \\ M \arrow[r, "\varphi"] & \bar M \end{tikzcd}$$

$\Phi(e_i) = e_i'$, $i = 1, \cdots, n$.

This leads to $\varphi : M \longrightarrow \bar M$ where $\varphi := \exp_{\bar p} \circ \Phi \circ (\exp_p)^{-1}$.

Notice that $\varphi$ is a <u>diffeomorphism</u>. So it remains to show

$$\varphi^* \bar g = g.$$

It's enough to show $\forall q \in M$ $\forall X \in T_qM$, ~~we have~~

$$\varphi^* \bar g (X, X) = g(X, X)$$

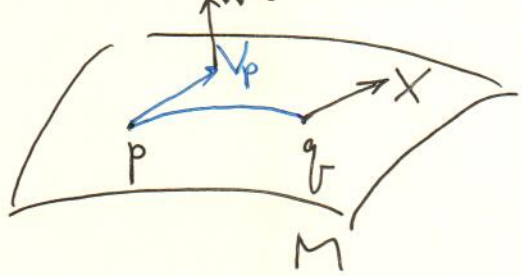
$$\shortparallel$$

$$\bar g (d\varphi_q(X), d\varphi_q(X))$$

i.e. the length of $d\varphi_q(X)$ equals the length of $X$, $\forall X \in T_qM, \forall q$.

Recall Lemma 1 (p. 202) tells,

there is a unique Jacobi field $U$ along the geodesic from $p$ to $q$ s.t. $U(0) = 0$ and $U(1) = X$.




So we can calculate $|X|^2 = g(X, X)$ by calculate the whole Jacobi field $U(t)$. In fact we can construct $U(t)$ explicitly.

Let $V_p \in T_pM$ be such that $\exp_p V_p = q$.

Let $W \in T_{V_p}(T_pM)$ be such that $(d\exp_p)_{(V_p)}(W) = X$.

Consider the variation

$$F(t, s) = \exp_p t(V_p + sW)$$

We ~~see~~ know the variational field $U(t) = \frac{\partial}{\partial s}\Big|_{s=0} F(t, s)$ is the Jacobi field with

$$U(0) = p, \quad U(1) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(V_p + sW) = (d\exp_p)_{(V_p)}(W) = X.$$

$$\dot U(0) = W.$$

Next, we show $\bar{g}(d\varphi_q(X), d\varphi_q(X))$ can also be calculated by computing a whole Jacobi field.

Consider $\bar{F}(t,s) = \exp_{\bar{p}} t\left(\Phi(V_p) + s\,\Phi(W)\right)$ (We identify $T_{V_p}(T_pM)$ with $T_pM$)

Similarly, the variation field $\left.\frac{\partial}{\partial s}\right|_{s=0} \bar{F}(t,s) := \bar{U}(t)$ is a Jacobi field with $\bar{U}(0)=0$, $\dot{\bar{U}}(0) = \Phi(W)$.

We claim $\bar{U}(1) = d\varphi_q(X)$. This is seen from

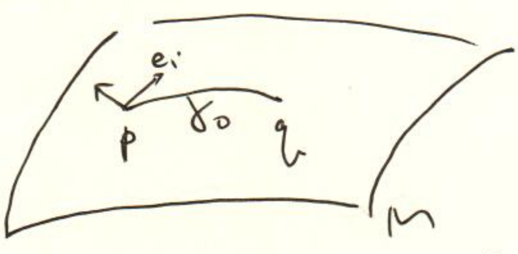
$$\begin{aligned}
\varphi \circ F(t,s) &= \exp_{\bar{p}} \circ \Phi \circ (\exp_p)^{-1} \circ \exp_p t(V_p + sW) \\
&= \exp_{\bar{p}} \circ \Phi\left(t(V_p + sW)\right) \\
&= \cancel{\Phi} \exp_{\bar{p}} t\left(\Phi(V_p) + s\,\Phi(W)\right) \\
&= \bar{F}(t,s).
\end{aligned}$$

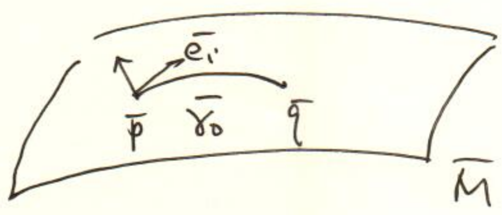
Hence
$$\underbrace{\left.\frac{\partial}{\partial s}\right|_{s=0} (\varphi \circ F(t,s))}_{\displaystyle = \, d\varphi_{F(t,s)} \circ \left.\frac{\partial}{\partial s}\right|_{s=0} F(t,s) = d\varphi_{F(t,s)}(U(t))} = \left.\frac{\partial}{\partial s}\right|_{s=0} \bar{F}(t,s) = \bar{U}(t)$$

In particular, $\bar{U}(1) = d\varphi_{F(1,0)}(U(1)) = d\varphi_q(X)$.

Hence, ~~we only nee~~ it remains to show

$$|\bar{U}(1)| = |U(1)|.$$




Pick parallel orthonormal vector fields $\{e_1(t), \cdots, e_n(t)\}$, $\{\bar{e}_1(t), \cdots, \bar{e}_n(t)\}$ along $\gamma_0$, $\bar{\gamma}_0$ respectively such that

$$e_i(0) = e_i, \quad \bar{e}_i(0) = \bar{e}_i.$$

Then $U(t) = f^i(t)\, e_i(t)$, $\quad \bar{U}(t) = \bar{f}^i(t)\, \bar{e}_i(t)$

for some fcts $f^i$, $\bar{f}^i$.

Solving Jacobi equations in $M$, $\bar{M}$ respectively:

in $M$: $\nabla_T\nabla_T U + R(U,T)T = 0$, $T := \dot{\gamma}_0(t)$.

~~we have~~

$$\Leftrightarrow$$

$$\ddot{f}^i(t)e_i(t) + R(f^j e_j, T)T = 0$$

$$\Leftrightarrow$$

$$\ddot{f}^i(t) + f^j\langle R(e_j,T)T, e_i\rangle = 0, \quad i=1,2,\cdots,n.$$

(sectional curvature $= c \Rightarrow \langle R(e_j,T)T,e_i\rangle = c\left(\delta_{ij}\langle T,T\rangle - \langle T,e_i\rangle\langle T,e_j\rangle\right)$)

$$\Leftrightarrow \quad \langle\dot{\gamma}_0(0),\dot{\gamma}_0(0)\rangle = \langle V_p,V_p\rangle \quad \text{parallelism}$$

$$\ddot{f}^i(t) + f^j\cdot c\,(\delta_{ij}\underbrace{\langle T,T\rangle}_{=\langle V_p,V_p\rangle} - \underbrace{\langle T,e_i\rangle}_{=\langle V_p,e_i\rangle}\underbrace{\langle T,e_j\rangle}_{=\langle V_p,e_j\rangle}) = 0, \quad i=1,2,\cdots,n.$$

~~Moreover~~ Recall we have further $U(0)=0$, $\dot{U}(0) = W$.

Hence $f^i$, $i=1,2,\cdots,n$ satisfies the following equations.

$$\begin{cases} \ddot{f}^i(t) + c f^j(t)\left(\delta_{ij}\langle \cancel{T}V_p, \cancel{T}V_p\rangle - \langle \cancel{T}V_p, e_i\rangle\langle \cancel{T}V_p, e_j\rangle\right) = 0, \quad i=1,2,\cdots,n \\ f^i(0) = 0 \\ \dot{f}^i(0) = \langle W, e_i\rangle \end{cases}$$

in $\bar{M}$: $\bar{f}^i$, $i=1,2,\cdots,n$ satisfies

$$\begin{cases} \ddot{\bar{f}}^i(t) + c\bar{f}^j\left(\delta_{ij}\langle\Phi(V_p),\Phi(V_p)\rangle - \langle\Phi(V_p),e_i'\rangle\langle\Phi(V_p),e_j'\rangle\right) = 0 \quad i=1,\cdots,n. \\ \bar{f}^i(0) = 0 \\ \dot{\bar{f}}^i(0) = \langle\Phi(W),e_i'\rangle \end{cases}$$

Since $\Phi : T_pM \to T_{\bar{p}}\bar{M}$ is an isometry, we have

$$\langle V_p,V_p\rangle = \langle\Phi(V_p),\Phi(V_p)\rangle, \quad \langle V_p,e_i\rangle = \langle\Phi(V_p),\Phi(e_i)\rangle = \langle\Phi(V_p),e_i'\rangle$$

$$\langle W,e_i\rangle = \langle\Phi(W),\Phi(e_i)\rangle = \langle\Phi(W),e_i'\rangle.$$

By the uniqueness of the solution, we have

$$f^i(t) = \bar{f}^i(t), \quad \forall t, \ \forall i\in\{1,2,\cdots,n\}$$

In particular,

$$|U(1)|^2 = \sum_i f^i(1)^2 = \sum_i \bar{f}^i(1)^2 = |\bar{U}^\Phi(1)|^2.$$

This proves the existence of ~~the~~ an isometry claimed in Theorem for the case $c = 0$ or $-1$.

~~Case 2 c=+1~~. The uniqueness of the isometry $\varphi$ follows from the following lemma.

Lemma 4. Let M be a connected Riemannian manifold and N be a Rie. mfld. Let $\varphi_1, \varphi_2 : M \to N$ be two locally isometry such that

$$\exists\, x \in M, \quad \varphi_1(x) = \varphi_2(x) = x' \in N.$$

$$(d\varphi_1)_x = (d\varphi_2)_x : T_xM \to T_{x'}N.$$

Then $\varphi_1 = \varphi_2$.

Proof: Define $A \subset M$ to be

$$A := \{ z \in M : \varphi_1(z) = \varphi_2(z),\ (d\varphi_1)_z = (d\varphi_2)_z \}$$

By assumption, $x \in A$, i.e. $A \neq \phi$. From the definition, A is closed. Next, we show A is open. Then since M is connected, we have $A = M$.

Suppose $z \in A$, then $z' = \varphi_1(z) = \varphi_2(z) \in N$. Choose $\delta > 0$ small enough, such that $\exp_z : B(0,\delta) \subset T_zM \to B_z(\delta) \subset M$ is a diffeomorphism and $\exp_{z'} : B(0,\delta) \subset T_{z'}N$ is defined.

$$\begin{array}{ccc} B(0,\delta) \subset T_zM & \xrightarrow{(d\varphi_i)_z} & B(0,\delta) \subset T_{z'}N \\ \downarrow \exp_z \text{ (diff.)} & & \downarrow \exp_{z'} \\ B_z(\delta) \subset M & \xrightarrow{\varphi_i} & B_{z'}(\delta) \subset N \end{array}$$

By similar argument in the proof of Lemma 3 on p. 216, we have $\varphi_i \circ \exp_z = \exp_{z'} \circ (d\varphi_i)_z$, $i = 1, 2$.

Notice that $\exp_z|_{B_z(\delta) \subset M}$ is invertible, we have

$$\varphi_i = \exp_{z'} \circ (d\varphi_i)_z \circ (\exp_z)^{-1}$$

Now we check $\forall\, y \in B_z(\delta)$,

$$\varphi_1(y) = \exp_{z'} \circ (d\varphi_1)_z \circ (\exp_z)^{-1}(y) = \exp_{z'} \circ (d\varphi_2)_z \circ \exp_z^{-1}(y) = \varphi_2(y).$$

and

$$(d\varphi_1)_y = (d\varphi_2)_y .$$

Therefore, we have $B_z(\delta) \subset A$, ~~$\Phi$~~ $\Rightarrow A$ is open $\square$

<u>Case 2</u> : $c = +1$. ~~W.l.o.g.~~ We can suppose $M = \mathbb{S}^n$. $\forall p \in \mathbb{S}^n$, any two geodesics from $p$ will meet together at its antipodal point $p'$. Therefore,

$$\exp_p^{-1} : S^n \setminus \{p'\} \to T_p S^n$$

is a well-defined smooth map.

$$\begin{tikzcd} T_p S^n \arrow[rr, "\Phi"] & & T_{\bar p} \bar M \arrow[d, "\exp_{\bar p}"] \\ S^n \setminus \{p'\} \arrow[u, "\exp_p^{-1}"] & & \bar M . \end{tikzcd}$$

where $\Phi$ is the isometry (of inner product spaces) with $\Phi(e_i) = e_i'$.

then ~~$\phi$~~ $\varphi : \exp_{\bar p} \circ \Phi \circ \exp_p^{-1}$ is a local isometry by the same argument as in <u>Case $c = 0$ or $-1$</u>. .

Next, we extend $\varphi$ to be defined on the whole $\mathbb{S}^n$. Pick any $z \in S^n \setminus \{p'\}$, $z \neq p$. let $z' = -z$ is the antipodal point of $z$.

Let $\varphi(z) = \bar z \in \bar M$, then $(d\varphi)_z : T_z \mathbb{S}^n \to T_{\bar z} \bar M$.

Define $\psi : \mathbb{S}^n \setminus \{z'\} \to \bar M$ as

$$\psi = \exp_{\bar z} \circ \cancel{\exp} (d\varphi)_z \circ \exp_z^{-1} .$$

Similar arguments tell that $\psi$ is also a locally isometry.

Consider the connected Ri. manifld $W := \mathbb{S}^n \setminus \{p', z'\}$. We have two local isometries

$$\varphi, \psi : W \to \bar M$$

Observe that

$$\psi(z) = \exp_{\bar z} \circ (d\varphi)_z \circ \exp_z^{-1}(z) = \bar z = \varphi(z)$$

$$\cancel{d\psi(z)=} (d\psi)_z = (d\varphi)_z$$

By Lemma 4, we have $\varphi = \psi|_{W = S^n \setminus \{p', z'\}}$.

Now define $\theta : S^n \to \bar M$ by

$$\theta(y) = \begin{cases} \varphi(y), & \text{if } y \in S^n \setminus \{p'\} \\ \psi(y), & \text{if } y \in S^n \setminus \{z'\} \end{cases}$$

This is a well defined $C^\infty$ map on $S^n$, and $\theta$ is a local isometry. By Lemma 3, we have $\theta$ is a ~~diffeomor~~ covering map. Since $\bar M$ is simply-connected, ~~we have~~ $\theta$ is a diffeomorphism and hence an isometry. Moreover $d\theta(e_i) = e_i'$. This proves the existence. The uniqueness follows again from Lemma 4. □.

Theorem 9 has very interesting consequences. When $M = \bar M$, we have:

<u>Corollary 1</u>: Let $M$ be a $n$-dim'l complete simply-connected Rie. mfld. Then $M$ is a space form <u>iff</u> $\forall p, \bar p \in M$, and any orthonormal basis $\{e_1, \cdots, e_n\}$, $\{\bar e_1, \cdots, \bar e_n\}$ of $T_pM$, $T_{\bar p}M$, respectively, there exists an isometry $\varphi : M \to M$ s.t.

$$\varphi(p) = \bar p, \quad d\varphi(e_i) = \bar e_i, \ \forall i.$$

<u>Definition</u> (homogeneous Rie. mfld) A Rie. mfld $(M, g)$ is called <u>homogeneous</u> (齐性) if $\forall p, q \in M$, there exists an isometry

$$\varphi : M \to M$$

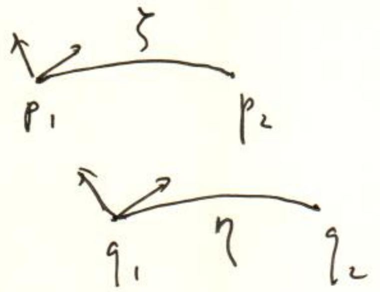
such that $\varphi(p) = q$.

$(M, g)$ is called two-point homogeneous, if for any two pairs of points $p_1, p_2$ and $q_1, q_2 \in M$ with $d(p_1, p_2) = d(q_1, q_2)$, there exists an isometry $\varphi : M \to M$ s.t.

$$\varphi(p_i) = q_i, \quad i = 1, 2.$$

Corollary 2. All ~~space~~ simply-connected space forms are two-point homogeneous. (两点齐性)

Proof: Let $d(p_1, p_2) = d(q_1, q_2) = \alpha$.




Let $\xi, \eta : [0, \alpha] \to M$ be two normal geodesics with

$$\xi(0) = p_1,\ \xi(\alpha) = p_2,\ \eta(0) = q_1,\ \eta(\alpha) = q_2$$

(The existence is guaranteed by completeness via Hopf-Rinow)

Pick orthonormal basis

$$\{e_1, \dots, e_n\} \quad \text{and} \quad \{e_1', \dots, e_n'\}$$

of $T_{p_1}M$ and $T_{q_1}M$, respectively, where $e_1 = \dot\xi(0)$, $e_1' = \dot\eta(0)$.

Thm 9 $\Rightarrow$ $\exists$ an isometry $\varphi : M \to M$ with

$$\varphi(p_1) = q_1, \quad \underline{d\varphi(e_i) = e_i'.\ \forall i.}$$

In particular, $d\varphi(e_1) = e_1'$

So $\varphi \circ \xi$ is a geodesic with

$$\varphi\circ\xi(0) = \varphi(p_1) = q_1, \quad (\varphi\circ\xi)^{\cdot}(0) = d\varphi(\dot\xi(0)) = d\varphi(e_1) = e_1'$$

$$= \eta(0) \qquad\qquad = \dot\eta(0)$$

Therefore $\varphi\circ\xi = \eta$.

In particular $\varphi(p_2) = \varphi(\xi(\alpha)) = \eta(\alpha) = q_2$. $\square$

## §9. Convexity: Another application of Cartan-Hadamard Theorem

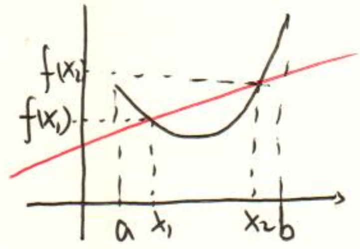
Convex functions and convex (sub)sets are important and useful concepts in analysis. We discuss these topics on Riemannian manifold in this section.

## What is a convex function?

Recall that we call a function $f: [a,b] \to \mathbb{R}$ to be <u>convex</u> if

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2), \quad \forall x_1, x_2 \in [a,b], \ \forall \lambda \in [0,1].$$

One can prove that a convex function must be Lipschitz continuous.




Recall for a $C^\infty$ function $f: [a,b] \to \mathbb{R}$, it is convex <u>iff</u> $f'' \ge 0$ on $[a,b]$.

This can be shown via its Taylor expansion.

( $$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots$$
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x^*)}{2}(x-x_0)^2$$
for some $x^*$ lying between $x_0$ and $x$.

($\Rightarrow$) Apply to the case $\begin{cases} x = x+h, \ x_0 = x, \\ x = x-h, \ x_0 = x \end{cases}$, we have

$$\left.\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} \\ \text{Convexity} \Rightarrow \quad f(x) &= f\left(\tfrac{1}{2}(x+h) + \tfrac{1}{2}(x-h)\right) \le \tfrac{1}{2}\left(f(x+h)+f(x-h)\right) \end{aligned}\right\} \Rightarrow f''(x) \ge 0$$

($\Leftarrow$). Apply to $\begin{matrix} x_0 = \lambda x_1 + (1-\lambda)x_2. \\ x = x_1 \end{matrix}$ gives

$$f(x_1) \ge f(\lambda x_1 + (1-\lambda)x_2) + f'(x_0)(1-\lambda)(x_1 - x_2) \qquad (1)$$

Apply to $x_0 = \lambda x_1 + (1-\lambda)x_2$, $x = x_2$ gives

$$f(x_2) \ge f(\lambda x_1 + (1-\lambda)x_2) + f'(x_0)\lambda(x_2 - x_1) \qquad (2)$$

Multiply (1) by $\lambda$, Multiply (2) by $(1-\lambda)$, and add them up provides

$$\lambda f(x_1) + (1-\lambda) f(x_2) \ge f(\lambda x_1 + (1-\lambda)x_2) \quad .)$$

We say a $C^\infty$ function $f$ is <u>strictly</u> convex, if $f'' > 0$.

Now consider a function $f: M \to \mathbb{R}$ where $M$ is a Riemannian manifold. A suitable definition of convexity is:

Definition (Convex function) We call a fct $f: M \to \mathbb{R}$ a convex function if for any geodesic $\gamma : [a,b] \to M$, $f\circ\gamma : [a,b] \to \mathbb{R}$ is convex, i.e. if $\forall\ t_1, t_2 \in [a,b]$, $\forall \lambda \in [0,1]$, it holds that.

$$f(\gamma(\lambda t_1 + (1-\lambda)t_2)) \le \lambda f(\gamma(t_1)) + (1-\lambda) f(\gamma(t_2)).$$

Proposition 9. A $C^\infty$ fct $f: M \to \mathbb{R}$ is ~~(strictly)~~ convex iff $(f\circ\gamma)'' \geq 0$ ~~$(>0)$~~ for any geodesic $\gamma$ in $M$, which is further equivalent to $\mathrm{Hess} f \geq 0$, i.e. $\mathrm{Hess} f$ is positive semidefinite.

Proof: $C^\infty$ $f: M \to \mathbb{R}$ is convex $\Leftrightarrow (f\circ\gamma)'' \geq 0$, $\forall \gamma$ follows from our previous discussions.

Notice further that for any $p \in M$, any $V_p \in T_pM$, letting $\gamma(t)$ be the geodesic with $\gamma(0) = p$, $\dot\gamma(0) = V_p$, we have

$$\mathrm{Hess} f(V_p, V_p) = \mathrm{Hess} f(\dot\gamma(t), \dot\gamma(t))\big|_{t=0}$$

$$= \nabla^2 f(\dot\gamma(t), \dot\gamma(t))\big|_{t=0} = \nabla(\nabla f)(\dot\gamma(t), \dot\gamma(t))\big|_{t=0}$$

$$= \nabla_{\dot\gamma(0)}(\nabla_{\dot\gamma(t)} f) - \underbrace{\nabla_{\nabla_{\dot\gamma}\dot\gamma} f}_{=0} = (f\circ\gamma)''.$$

Hence $(f\circ\gamma)'' \geq 0, \forall \gamma \Leftrightarrow \mathrm{Hess} f(V_p, V_p) \geq 0$, $\forall p$, $\forall V_p \in T_pM$. $\square$

~~Next, Let us consider a particular function.~~

We say a $C^\infty$ function $f: M \to \mathbb{R}$ to be a strictly convex function if $\mathrm{Hess} f > 0$.

Next, Let us consider a particular function on $M$.

Given a fixed point $O \in M$, consider the function

$$\rho(\cdot) := d(\cdot, O) : M \to \mathbb{R}$$

Theorem 10: Let $M$ be a complete, simply-connected Riemannian geometry with nonpositive sectional curvature. Let $O \in M$. Then the function $\rho^2 : M \to \mathbb{R}$ is $C^\infty$ and strictly convex.

Example: In $\mathbb{R}^n$ with the canonical Euclidean metric, (let $O = 0 \in \mathbb{R}^n$) we compute

$$\mathrm{Hess}\,\rho^2(X,X) = X^i X^j \frac{\partial^2}{\partial x^i \partial x^j}\rho^2(x) = X^i X^j \frac{\partial^2}{\partial x^i \partial x^j}\sum_k (x^k)^2$$

$$= X^i X^j \cdot 2\sum_k \delta_{kj}\delta_{ki} = 2\sum_k (X^k)^2 = 2|X|^2.$$

First, we observe, without any curvature restrictions, $\rho^2$ is always "locally" strictly convex.

Lemma 5: Let $M$ be a Rie. mfld, and $O \in M$. Then there exists a neighbor $\mathcal{U}_0$ of $O$ s.t. $\rho^2$ is smooth and strictly convex in $\mathcal{U}_0$.

Proof: Let $(U, x^1, \ldots, x^n)$ be a normal coordinate neighborhood of $O \in M$, such that $x^i(O) = 0$. Then

$$\rho^2(x) = \sum_{i=1}^n (x^i)^2 \quad \forall x \in U.$$

Recall any geodesic $\gamma$ with $\gamma(0) = O$, $\dot\gamma(0) = V \in T_O M$, ($=(V^1, \ldots, V^n)$) can be written as $\gamma(t) := (x^1(t), \ldots, x^n(t))$ where

$$x^i(t) = V^i t, \quad \rho^2 \circ \gamma = \sum_{i=1}^n t^2 (V^i)^2.$$

Hence $\mathrm{Hess}\,\rho^2(V,V) = (\rho^2 \circ \gamma)'' = 2\sum_{i=1}^n (V^i)^2 > 0.$

Therefore, there exists a neighborhood of $O$, $\mathcal{U}_0 \subset U$, s.t. $\mathrm{Hess}\,\rho^2$ is positive definite on $\mathcal{U}_0$. $\square$

But for "global" results, we need curvature restriction.

Let us ~~first~~ recall the first ~~Second~~ variation formula for length fct'ls.

(Thm 2. III §5, p.96 and Exercise 8)

<u>Lemma 6</u>: Let $\gamma : [a,b] \to M$ be a normal geodesic, and

$$F : [a,b] \times (-\varepsilon, \varepsilon) \to M$$

be a variation of $\gamma$ with variational field $V(t)$, $t \in [a,b]$. Then

$$L'(0) := \frac{d}{ds}\Big|_{s=0} L(s) = \langle V(t), \dot\gamma(t) \rangle\Big|_a^b - \int_a^b \langle V(t), \underbrace{\nabla_{\dot\gamma}\dot\gamma}_{=0} \rangle \, dt.$$

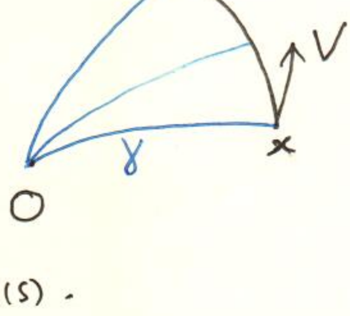
$$L''(0) := \frac{d^2}{ds^2}\Big|_{s=0} L(s) = \langle \nabla_V V, \dot\gamma \rangle\Big|_a^b + \int_a^b \langle \nabla_{\dot\gamma} V^\perp, \nabla_{\dot\gamma} V^\perp \rangle - \langle R(V^\perp, \dot\gamma)\dot\gamma, V^\perp \rangle \, dt$$

where $V^\perp := V - \langle V, \dot\gamma \rangle \dot\gamma$.

<u>Proof of Theorem 10</u>: By Cartan-Hadamard theorem and the definition of $\exp_O$, we have

$$\forall x \in M;\ \rho^2(x) = g(\underbrace{\exp_O^{-1}(x)}_{\in T_OM}, \exp_O^{-1}(x)) = |\exp_O^{-1}(x)|^2.$$

is a $C^\infty$ function on $M$. By Lemma 5, it remains to show that $\mathrm{Hess}\,\rho^2(V,V) > 0$ for any ~~$0 \neq V \in T$~~ $x \in M$, any $0 \neq V \in T_xM$




Let $\zeta : [0,\varepsilon] \to M$ be the geodesic of $M$ with $\zeta(0) = x$, $\dot\zeta(0) = V$.

Let $\gamma_s$, $s \in [0,\varepsilon]$ be the ~~∇~~ geodesics from $O$ to ~~$\gamma$~~ $\zeta(s)$.

In our setting, such geodesic is unique.

Let us parametrize $\gamma_s$ to be $\gamma_s : [0, r] \to M$ where $r = \rho(x)$.

Hence $\gamma := \gamma_0$ is a normal geodesic.

Hence, we have the following variation

$$F : [0,r] \times [0,\varepsilon] \to M$$

$$F(t,s) := \gamma_s(t).$$

Notice that the corresponding variational field $V(t)$, satisfies

$$V(0)=0, \quad V(r) = \frac{\partial}{\partial s}\Big|_{s=0} F \Big|_{t=r} = \frac{\partial}{\partial s}\Big|_{s=0} F(r,s)$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \zeta(s) = \dot{\zeta}(0) = V.$$

Now we compute

$$\begin{aligned} \mathrm{Hess}\,\rho^2(V,V) &= (\rho^2\circ\zeta)''(0) \\ &= \frac{d^2}{ds^2}\left(\rho^2\circ\zeta(s)\right)\Big|_{s=0} \\ &= 2\rho(\zeta(s))\frac{d^2}{ds^2}\rho(\zeta(s))\Big|_{s=0} + 2\left(\frac{d}{ds}\rho(\zeta(s))\Big|_{s=0}\right)^2 \\ &= 2r\,\frac{d^2}{ds^2}\Big|_{s=0}\rho(\zeta(s)) + 2\left(\frac{d}{ds}\Big|_{s=0}\rho(\zeta(s))\right)^2. \end{aligned}$$

Recall by Cartan-Hadamard theorem,

$$\rho(\zeta(s)) := d(\zeta(s), O) = L(\gamma_s).$$

Therefore Lemma 6 tells us

$$\mathrm{Hess}\,\rho^2(V,V) = 2r.L''(0) + 2(L'(0))^2 = 2r\,L''(0) + 2\left(\langle V, \dot{\gamma}(r)\rangle\right)^2$$

~~$= 2r$~~. Notice that we have $\nabla_V V\big|_{t=0 \text{ or } r} = 0$, and hence

$$L''(0) = \int_0^r \Big(\langle \nabla_T V^\perp, \nabla_T V^\perp\rangle \underbrace{- \langle R(V^\perp,\dot\gamma)\dot\gamma, V^\perp\rangle}_{\geq 0}\Big)dt$$

$$\geq \int_0^r \langle \nabla_T V^\perp, \nabla_T V^\perp\rangle\, dt.$$

That is,

$$\mathrm{Hess}\,\rho^2(V,V) \geq 2r\int_a^b \langle \nabla_T V^\perp, \nabla_T V^\perp\rangle dt + 2\left(\langle V, \dot\gamma(r)\rangle\right)^2$$

(1) If $\langle V, \dot\gamma(r)\rangle \neq 0$, we obtain $\mathrm{Hess}\,\rho^2(V,V) \geq 2\langle V, \dot\gamma(r)\rangle^2 > 0$.

(2) Otherwise if $\langle V, \dot\gamma(r)\rangle = 0$. Since $V$ is a Jacobi field and

$\langle V, \dot\gamma(0)\rangle = \langle 0, \dot\gamma(0)\rangle = 0$, Prop 8 (p.194) tells

$\langle V(t), \dot\gamma(t)\rangle = 0$, $\forall t \in [0,r]$. Therefore, $V(t) = V^\perp(t)$.

We observe that $\nabla_T V \not\equiv 0$. Since otherwise $V(t)$ is parallel along $\gamma$, which contradicts to the fact $V(0) = 0$, $V(r) = V \neq 0$.

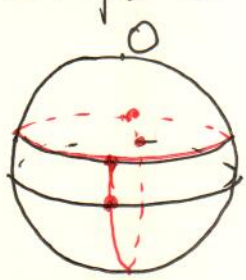
That is, $\text{Hess}\,\rho^2(V,V) \geq \int_0^r \langle \nabla_T V^\perp, \nabla_T V^\perp\rangle\, dt > 0$. □

Definition (Convex and totally convex subset of M). Let M be a Rie. mfld. A subset $\Omega \subset M$ is called convex, if whenever $p, q \in \Omega$ and $\gamma$ is a minimizing geodesic from p to q, then $\gamma \subset \Omega$. $\Omega$ is called totally convex if whenever $p, q \in \Omega$ and $\gamma$ is a geodesic from p to q, then $\gamma \subset \Omega$.

Recall by Cartan-Hadamard theorem, on a complete simply-connected Rie. mfld with nonpositive sectional curvature, any geodesic is minimizing, and, hence, any convex subset is totally convex. However, these two concepts do have differences.

Example On $S^2 \subset \mathbb{R}^3$ the unit sphere,

$\{p \in S^2 \mid d(p, O) < r\}$ where $r \leq \frac{\pi}{2}$,




is convex, but is not totally convex.

$\{p \in S^2 \mid d(p, O) \leq \frac{\pi}{2}\}$ is not convex.

Convex functions and convex subsets are related by the following result.

Proposition 10: Let $\tau : M \to \mathbb{R}$ be a convex function on a complete Rie. mfld. M. Then the sub-level set

$$M_c := \{ x \in M : \tau(x) < c \}$$

is totally convex.

Proof: $\forall\, p, q \in M_c$, any any geodesic $\gamma : [a,b] \to M$ from $p$ to $q$, we have $(\tau\circ\gamma)'' \geq 0$. Therefore $\tau\circ\gamma : [a,b] \to \mathbb{R}$ attains its maximum at the two ends. Hence

$$\tau\circ\gamma(t) \leq \max\{\tau\circ\gamma(a), \tau\circ\gamma(b)\} = \max\{\tau(p), \tau(q)\} < c.$$

That is $\gamma \subset M_c$. $\square$

Therefore, Theorem 10 tells that any (open or closed) geodesic balls

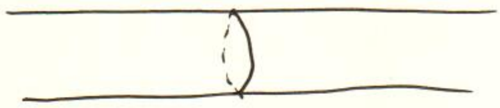
$$\{ x \in M : d(x, O) \underset{(\leq)}{<} r \}$$

is totall convex on a complete & simply-connected Rie. mfld with nonpositive sectional curvature. In particular, every point is totally convex (i.e. no nontrivial geodesic $\gamma : [a,b] \to M$ with $\gamma(a) = \gamma(b) = x$ exists).

Proper totally convex sets ($\Omega \neq M$) do not exists in many manifolds. Existence of such kind of subsets has significant topological implications.

Theorem (The Soul Theorem, Cheeger - Gromoll 1972) If $(M, g)$ is a complete non-compact Riemannian manifold with nonnegative sectional curvature, then $M$ contains a closed totally convex submanifold $S$, such that $M$ is diffeomorphic to the normal bundle over $S$.

$S$ is called a soul of $M$.



$\{(x,y,z) \mid \cancel{x^2+y^2=1}, z = c\}$ is a soul of the cylinder.

We also explain the geometric meaning of the local result Lemma 5.

Theorem 11 (Whitehead 1932). Let $(M,g)$ be a Riemannian mfld. Any $p \in M$ has a convex neighborhood. ~~In fact~~

Proof: Recall for any $p \in M$, there exists a totally normal neighborhood, that is, a neighborhood $W \ni p$ and a number $\delta > 0$ such that any two $q_1, q_2 \in W$ can be joined by a unique minimizing geodesic. However, such a geodesic may not lie completely in $W$.

By Lemma 5, there exists a neighborhood $\mathcal{U}_p$ s.t. $d^2(\cdot, p)$ is strictly convex in $\mathcal{U}_p$.


Pick $r$ small enough s.t.

$$B_p(r) := \{q \in M, d(q,p) < r\} \subset \mathcal{U}_p \cap W.$$

The proof of Proposition 10 tells $B_p(r)$ is convex. $\square$