(VI) Comparison Theorems.

Recall in our discussions about Cartan-Hadamard and Bonnet-Myers theorems, we, in fact, have model spaces in mind.

(1) Cartan-Hadamard: use $\mathbb{R}^n$ as a model, and replace the "zero curvature" of $\mathbb{R}^n$ by "cur ≤ 0".

(2) Bonnet-Myers: use $\mathbb{S}^n$ as a model, and replace "Ricci curvature $= n-1$" of $\mathbb{S}^n$ by "Ricci curv. $\geq (n-1)$".

In this chapter, we aim at establishing quantitative comparison result with model spaces.

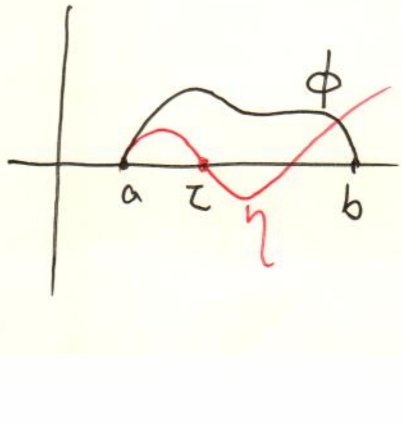
§1. Sturm Comparison Theorem. [Spivak IV, Chap. 8, 15-17]

We start from a pure analysis result of Sturm.

Theorem 1. (Sturm) Let f and h be two continuous functions satisfying $f(t) \leq h(t)$ for all t in an interval I, and let ϕ and η be two functions satisfying the differential equations:

$$\begin{cases} \phi'' + f\phi = 0 & (1) \\ \eta'' + h\eta = 0 & (2) \end{cases} \quad \text{on } I.$$

Assume that ϕ is not the zero fct and let $a, b \in I$ be two consecutive zeros of ϕ. Then:



(1) The function η must have a zero in (a,b), unless $f = h$ everywhere on $[a,b]$ and η is a constant multiple of ϕ on $[a,b]$

(2) Suppose that $\eta(a)=0$, and also $\eta'(a)=\phi'(a)>0$. If τ is the smallest zero of η in (a,b), then

$$\phi(t) \geq \eta(t) \quad \text{for} \quad a \leq t \leq \tau$$

and equality holds for some t only if $f=h$ on $[a,t]$.

<u>Remark</u>: The restriction $\eta'(a)=\phi'(a)>0$ in (2) can be achieved by choosing a suitable multiple of η, and changing ϕ to $-\phi$ if necessary.

<u>Proof</u>: Equations (1) and (2) gives

$$\eta\phi'' - \phi\eta'' = (h-f)\,\phi\eta \,. \qquad (3)$$

Suppose that η were nowhere zero on (a,b). W.o.l.g., we can assume

$$\eta,\ \phi > 0 \ \text{on}\ (a,b) \qquad (4)$$

Then (3) gives

$$\eta\phi'' - \phi\eta'' = (h-f)\,\phi\eta \geq 0 \,.$$

Therefore

$$0 \leq \int_a^b (\eta\phi'' - \phi\eta'') = \int_a^b (\eta\phi' - \phi\eta')'$$

$$= \eta(b)\phi'(b) - \eta(a)\phi'(a) - \big(\underbrace{\phi(b)}_{=0}\eta'(b) - \underbrace{\phi(a)}_{=0}\eta'(a)\big)$$

$$= \eta(b)\phi'(b) - \eta(a)\phi'(a). \qquad (5)$$

On the other hand, (4) implies

$$\left.\begin{matrix} \phi'(a) > 0, & \phi'(b) < 0 \\ \eta(a) \geq 0, & \eta(b) \geq 0 \end{matrix}\right\} \Rightarrow \eta(b)\phi'(b) - \eta(a)\phi'(a) \leq 0. \qquad (6)$$

If $f \neq h$, we have

$$0 < \int_a^b (\eta\phi'' - \phi\eta'')$$

which is a contradiction to (6). Hence η must have a zero on (a,b).

If $f = h$ on $[a,b]$, then (5) + (6) $\Rightarrow$

$$0 = \underset{\geq 0}{\eta(b)} \underbrace{\phi'(b)}_{<0} - \underset{\geq 0}{\eta(a)} \underbrace{\phi'(a)}_{>0} \Rightarrow \eta(a) = \eta(b) = 0.$$

Now ϕ and η satisfy the same equation

$$\begin{cases} \phi'' + f\phi = 0 \\ \eta'' + f\eta = 0 \end{cases} \quad \text{on } [a,b]$$

and $\phi(a) = \eta(a) = 0$. The solution η must be a constant multiple of ϕ on $[a,b]$.

Next suppose $\eta(a) = 0$, $\eta'(a) = \phi'(a) > 0$. (recall $\phi(a) = 0$). Let τ be the smallest zero of η ~~in~~ in $(a,b]$. Then $\phi > 0, \eta > 0$ on (a,τ).

Hence
$$\phi''\eta - \eta''\phi = (h - f)\phi\eta \geq 0 \text{ on } (a,\tau)$$
$$\overset{\|}{(\phi'\eta - \eta'\phi)'}.$$

Recall $\phi'(a)\eta(a) - \eta'(a)\phi(a) = 0$, this implies
$$\phi'\eta - \eta'\phi \geq 0 \quad \text{on } (a,\tau).$$

Since $\eta > 0$ on (a,τ), we obtain
$$\frac{\phi'\eta - \eta'\phi}{\eta^2} = \left(\frac{\phi}{\eta}\right)' \geq 0 \quad \text{on } (a,\tau).$$

L'Hôpital's Rule

But $\displaystyle\lim_{t\to a} \frac{\phi(t)}{\eta(t)} = \lim_{t\to a} \frac{\phi'(t)}{\eta'(t)} = 1$. Assumption.

Therefore, $\dfrac{\phi}{\eta} \geq 1$ on (a,τ).

This proves $\phi(t) \geq \eta(t)$ for $a \leq t \leq \tau$.

If $\phi(t) = \eta(t)$ for some t, then $\dfrac{\phi}{\eta} \equiv 1$ on (a,t)

Hence $\left(\frac{\phi}{\eta}\right)' = 0 \Rightarrow \phi'\eta - \eta'\phi = 0$ on $(a,t) \Rightarrow \phi''\eta - \eta''\phi = 0$ on (a,t)

$\Rightarrow f = h$ on (a,t). By continuity, $f = h$ on $[a,t]$. $\square$

Geometric translations.

<u>Theorem 2</u> (Bonnet, 1855) Let M be a surface, and $\gamma : [0, L] \to M$ be a normal geodesic. Let $k > 0$.

(1) If $K(p) \leq k$ for all $p = \gamma(t)$, and γ has length $L < \frac{\pi}{\sqrt{k}}$ then γ contains <u>no</u> conjugate point.

(2) If $K(p) \geq k$ for all $p = \gamma(t)$, and γ has length $L > \frac{\pi}{\sqrt{k}}$, then there is a point $\tau \in (0, L)$ conjugate to 0, and therefore γ is not of minimal length.

<u>Proof</u>: Let Y be a unit parallel vector field along γ with $\langle Y, \dot\gamma \rangle = 0$, $\forall t$.

γ Y

Any normal Jacobi field U can be written as

$$U = \phi Y \quad \text{for some fct } \phi.$$

Jacobi equation $\nabla_T \nabla_T U + R(U,T)T = 0$ implies

$$\phi''(t) + K(Y,T)\,\phi(t) = 0. \quad (1)$$

The corresponding discussion on constant curved surfaces (model spaces) gives

$$\eta''(t) + k\,\eta(t) = 0. \quad (2)$$

with a solution $\eta(t) = \sin(\sqrt{k}\,t)$. Note 0 and $\frac{\pi}{\sqrt{k}}$ are two consecutive zeros of η.

(1) $K(Y,T) \leq k$, $\forall t$. thm 1 (1) implies (1) cannot have a solution ϕ vanish at 0 and at $L < \frac{\pi}{\sqrt{k}}$. Since otherwise, $\sin(\sqrt{k}\,t)$ has to vanish at somepoint $\in (0, L)$, which is false.

(2) $K(Y,T) \geq k$, $\forall t$. Thm 1 (1) implies any Jacobi field ϕY must has a zero on $(0, \frac{\pi}{\sqrt{k}}) \subset (0, L)$ So if we choose any nonzero Jacobi field ϕY along γ with $\eta(0)=0$, this Jacobi field will also vanish at some $\tau \in (0, L)$; thus τ is conjugate to 0. $\square$

~~Of course~~. Thm 2 (2) is the result which Bonnet used ~~to~~ show his diameter estimate.

From the above proof, we observe the following facts: The Jacobi field $U = \phi Y$ where Y is a unit parallel vector field along γ with $\langle Y, \dot\gamma \rangle = 0$, we have

$$\phi''(t) + K(\gamma(t))\phi(t) = 0.$$

$$\phi(a) = 0 \iff U(a) = 0$$

$$\phi'(a) = |\dot U(a)| \qquad |\phi(a)| = |U(a)|.$$

So Sturm comparison theorem (2) can be translated as:

- Given two ~~Riem. mani~~ surfaces M_0 and $\bar M$ let $\gamma_0 : [a,b] \to M_0$, $\bar\gamma : [a,b] \to \bar M$ be two normal geodesics such that

$$K(\gamma_0(t)) \leq \bar K(\bar\gamma(t))$$

let $\tau \in [a,b]$ s.t. $\gamma, \bar\gamma$ have no point in $[a,\tau]$ conjugate to $\gamma(a), \bar\gamma(a)$ resp.

Let $U, \bar U$ be normal Jacobi fields along $\gamma, \bar\gamma$ resp. with

$$U(a) = \bar U(a) = 0$$
$$|\dot U(a)| = |\dot{\bar U}(a)|$$

Then $|U(t)| \geq |\bar U(t)|$, for $a \leq t \leq \tau$.

and "=" holds for some t only if $K \circ \gamma = \bar K \circ \bar\gamma$ on $[a,t]$.

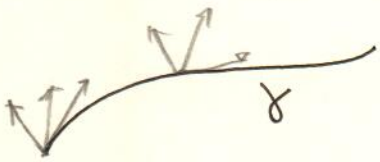
Remark. The above "comparison of Jacobi fields" implies Bonnet Theorem 2 by choose one of $M, \bar{M}$ to be the sphere $S^2(\frac{1}{\sqrt{k}})$. In fact in Sturm comparison theorem 1, $(2) \Rightarrow (1)$ when $\eta(a)=0$ is the case.

§2. Morse – Schoenberg Comparison and Rauch Comparison Theorems.

[Spivak IV, Chap. 8, 18–23]

It is natural to ask for higher-dimensional generalizations of ~~those~~ the geometric translations of Sturm Comparison Theorem.

Let (M,g) be a n-dimensional Rie. mfld



$\gamma : [a,b] \to M$ be a normal geodesic

Now a normal Jacobi field U along γ cannot always be written as ϕY where Y is a unit normal parallel vector field along γ.

In fact, Let $\{Y_1, \cdots, Y_n\}$ be an orthonormal parallel vector field along γ with $\dot{\gamma}(t) = Y_1(t)$. Then a normal Jacobi field U along γ can be written as

$$U(t) = \sum_{i=2}^{n} \phi_i(t) Y_i(t)$$

Jacobi equation $\Rightarrow \sum_{i=2}^{n} \phi_i''(t) Y_i(t) + \sum_{i=2}^{n} \phi_i(t) R(Y_i(t), T)T. = 0$

$$\Rightarrow \quad \phi_j''(t) + \sum_{i=2}^{n} \phi_i(t) \langle R(Y_i, T)T, Y_j \rangle = 0. \quad j=2, \cdots, n$$

This system of equations do not involve the sectional curvature directly.

$$\frac{d^2}{dt^2}(\phi_2(t), \cdots, \phi_n(t)) + (\phi_2(t), \cdots, \phi_n(t)) \begin{pmatrix} \langle R(Y_2,T)T, Y_2 \rangle & \cdots & \langle R(Y_2,T)T, Y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle R(Y_n,T)T, Y_2 \rangle & \cdots & \langle R(Y_n,T)T, Y_n \rangle \end{pmatrix}$$

$= 0.$

Recall the space of normal Jacobi fields along γ vanishing at $t=a$ is of dimension $n-1$. We actually have to solve the following to solve the equation to ~~find an~~ compute ~~a~~ Jacobi fields:

$$\begin{cases} \dfrac{d^2}{dt^2} A + A R = 0 \\ A(0) = 0 \quad \dfrac{dA}{dt}(0) = Id_{n-1} \end{cases} \qquad (*)$$

where $R = (\langle R(Y_i, T)T, Y_j \rangle)_{i,j}$ is symmetric.

We will not discuss the ~~direct~~ generalization of Sturm comparison theorem to the equation $(*)$, but instead, will discuss the ~~ge~~ generalization of its geometric translations. ~~~~ These two are different aspect of the same result. (See [WSY, Chap. 8, Appendix])

From the geometric viewpoint, we are going to compare the Jacobi fields along geodesics in two Rie. manifolds, whose sectional curvatures satisfy certain comparison estimate. For that purpose, we need "move" a vector field along ~~a~~ a geodesic γ in a Rie. mfld M to ~~an~~ a geodesic $\bar{\gamma}$ in another Rie. mfld $\bar{M}$.

<u>Lemma 1</u>. Let (M, g), $(\bar{M}, \bar{g})$ be two Rie. mfld of the same dimension n, and let $\gamma\,(\bar{\gamma}) : [a,b] \to M\ (\bar{M})$ be ~~two~~ a normal geodesic in $M\ (\bar{M})$. Then there is a vector space isomorphism

$$\Phi : \{\text{piecewise } C^\infty \text{ vector fields along } \gamma\}$$
$$\to \{\text{piecewise } C^\infty \text{ vector fields along } \gamma_2\}$$

such that for all $t \in [a,b]$, we have for any piecewise C^∞ vector field X along γ.

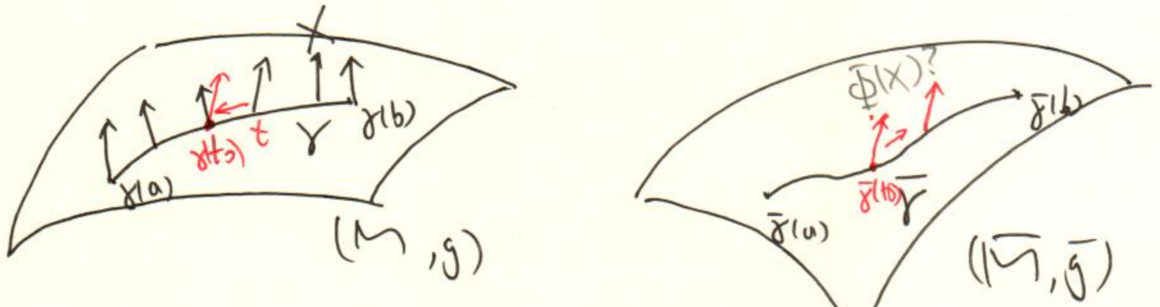
(1) If $\nabla_T X := \tilde{\nabla}_{\frac{\partial}{\partial t}} X$ is continuous at t, then $\nabla_{\bar{T}} \Phi(X) := \tilde{\nabla}_{\frac{\partial}{\partial t}} \Phi(X)$ is continuous at t.

(2) $\langle X(t), \underset{\overset{\shortparallel}{T(t)}}{\dot{\gamma}(t)} \rangle_g = \langle \Phi(X)(t), \underset{\overset{\shortparallel}{\bar{T}(t)}}{\dot{\bar{\gamma}}(t)} \rangle_{\bar{g}}$

(3) $|X(t)|_g = |\Phi(X)(t)|_{\bar{g}}$, where $|X(t)|_g = \sqrt{g(X(t), X(t))}$.

(4) $|\nabla_T X(t)|_g = |\nabla_{\bar{T}} \Phi(X)(t)|_{\bar{g}}$, it being understood that this equation refers to left and right hand limit at discontinuity points.

Proof:



What ~~wil~~ could be a natural choice of such a Φ?

An isomorphism between $T_{\gamma(t_0)}M$ and $T_{\bar{\gamma}(t_0)}\bar{M}$ for a fixed point t_0 is easy: Pick $\phi_{t_0}: T_{\gamma(t_0)}M \to T_{\bar{\gamma}(t_0)}\bar{M}$ to be one isomorphism which preserve the inner products given by g and $\bar{g}$ respectively.

How to extend it? For any $t \in [a,b]$, we define

$$\phi_t : T_{\gamma(t)}M \to T_{\bar{\gamma}(t)}\bar{M}$$

$$V_t \xmapsto{\text{parallel transport along } \gamma \text{ to } \gamma(t_0)} P_{\gamma,t,t_0}(V_t)$$

$$\xmapsto{\phi_{t_0}} \phi_{t_0}(P_{\gamma,t,t_0}(V_t)) \longmapsto P_{\bar{\gamma},t_0,t}(\phi_{t_0}(P_{\gamma,t,t_0}(V_t)))$$

Then define $\Phi(X)(t) = \phi_t(X(t))$.

Next, we give Φ an explicit expression. Let $Y_1, \cdots, Y_n$ be parallel, everywhere orthonormal vector fields along γ with $Y_1(t_0) = \dot{\gamma}(t_0)$. ~~Let $Z_1, \cdots, Z_n$ be parallel~~ Let $\bar{Y}_1, \cdots, \bar{Y}_n$ be parallel, everywhere orthonormal vector fields along $\bar{\gamma}$ with $\bar{Y}_1(t_0) = \dot{\bar{\gamma}}(t_0)$.

~~The~~ A piecewise C^∞ vector field $X(t)$ along γ can be written as

$$X(t) = \sum_{i=1}^{n} f_i(t) Y_i(t)$$

for certain functions $f_i : [a,b] \to \mathbb{R}$, ~~then~~ ϕ_{t_0} is chosen s.t.

$$\cancel{\Phi(X)(t) = \phi_t(X(t)) =}$$

$$\phi_{t_0}(Y_i(t_0)) = \bar{Y}_i(t_0), \quad i = 1, 2, \cdots, n.$$

Then $\Phi(X)(t) = \phi_t(X(t)) = \sum_{i=1}^{n} f_i(t) \bar{Y}_i(t)$.

This shows $\Phi(X)$ is C^∞ everywhere that X is, and that

$$\langle X(t), \dot{\gamma}(t) \rangle = f_1(t) = \langle \Phi(X)(t), \dot{\bar{\gamma}}(t) \rangle$$

$$|X(t)|_g^2 = \sum_{i=1}^{n} f_i^2(t) = |\Phi(X)(t)|_{\bar{g}}^2$$

$$|\nabla_T X(t)|_g^2 = \sum_{i=1}^{n} (f_i'(t))^2 = |\nabla_{\bar{T}} \Phi(X)(t)|_{\bar{g}}^2. \quad \square$$

<u>Theorem 3</u>: Let $M, \bar{M}$ be two Rie. mfld of the same dimension n, and let $\gamma(\bar{\gamma}) : [a,b] \to M(\bar{M})$ be a normal geodesic. For each $t \in [a,b]$, suppose that for all 2-dim. sections $\Pi_{\gamma(t)} \subset T_{\gamma(t)} M$, and all 2-dim sections $\bar{\Pi}_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)} \bar{M}$, the sectional curvatures satisfy

$$K(\Pi_{\gamma(t)}) \le \bar{K}(\bar{\Pi}_{\bar{\gamma}(t)}).$$

Then we have

$$\operatorname{ind}(\gamma) \le \operatorname{ind}(\bar{\gamma}).$$

piecewise C^∞ vector fields along γ with $W(a) = W(b) = 0$.

In particular, if $I(W,W) < 0$ for some ~~piecewise~~ $W \in \mathcal{V}_0(a,b)$, then also $\bar{I}(\bar{W}, \bar{W}) < 0$ for some $\bar{W} \in \bar{\mathcal{V}}_0(a,b)$.

Proof: Let $W \in \mathcal{V}_0(a,b)$, recall that

$$I(W,W) = \int_a^b \{ \langle \nabla_T W, \nabla_T W \rangle - \langle R(W,T)T, W \rangle \} dt$$

Let Φ be constructed as in Lemma 1. Then $\Phi(W) \in \overline{\mathcal{V}}_0(a,b)$.

and $\langle \nabla_T W, \nabla_T W \rangle_g = \langle \nabla_{\bar{T}} \Phi(W), \nabla_{\bar{T}} \Phi(W) \rangle_{\bar{g}}$

$$\begin{aligned}\langle R(W,T)T\,W \rangle_g &= K(W,T) \cdot \left(\langle W,W \rangle \langle T,T \rangle - \langle W,T \rangle^2 \right) \\ &\leq \bar{K}(\Phi(W),\bar{T}) \left(\langle \Phi(W),\Phi(W) \rangle \langle \bar{T},\bar{T} \rangle - \langle \Phi(W), \bar{T} \rangle^2 \right) \\ &= \langle \bar{R}(\Phi(W),\bar{T})\bar{T}, \Phi(W) \rangle\end{aligned}$$

That is, $I(W,W) \geq \bar{I}(\Phi(W), \Phi(W))$.

So if $\mathcal{A} \subset \mathcal{V}_0(a,b)$ is a subspace on which I is negative definite, then $\Phi(\mathcal{A}) \subset \overline{\mathcal{V}}_0(a,b)$ is a subspace of the same dimension on which $\bar{I}$ is again negative definite. By definition, this means $\operatorname{ind}(\gamma) \leq \operatorname{ind}(\bar{\gamma})$. □

Corollary 1 (The Morse-Schoenberg Comparison theorem).
Let (M,g) be a Rie. mfld of dim. n, and let $\gamma : [0,L] \to M$ be a normal geodesic. Let $k>0$.

(1) If $K(\Pi_{\gamma(t)}) \leq k$ for all $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, and γ has length $L < \frac{\pi}{\sqrt{k}}$, then ~~γ contains no~~ $\operatorname{ind}(\gamma) = 0$, and γ contains no conjugate point.

(2). If $K(\Pi_{\gamma(t)}) \geq k$ for all $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, and γ has length $L > \frac{\pi}{\sqrt{k}}$, then there is a point $\tau \in (0,L)$ conjugate to 0, and γ is not of minimal length.

Remark: (a) This is a high-dimensional generalization of Bonnet Theorem 2 (p.238)

(b). (1) is a generalization of Prop 9 (p.209), which asserts M contains no conjugate points if $\sec \leq 0$.

Proof : (1) We apply Theorem 3 to $M = M$, $\bar{M} = \mathbb{S}^n(\frac{1}{\sqrt{k}})$

Choosing $\gamma : [0,L] \to M$, $\bar{\gamma} : [0,L] \to \mathbb{S}^n(\frac{1}{\sqrt{k}})$ to be normal geodesic. We have

$$\mathrm{ind}(\gamma) \le \mathrm{ind}(\bar{\gamma}).$$

Now $\mathrm{ind}(\bar{\gamma}) = 0$ since $\bar{\gamma}$ contains no conjugate points. (Thm 6, p 202, or the Morse Index Theorem) Therefore $\mathrm{ind}(\gamma) = 0$ which implies γ contains no conjugate point. (by Thm 6, p.202, again).

(2) Similar argument. Recall (2) has already been proved when we discussed Bonnet - Myers Theorem. (p.156). Now it is a good chance to understand the proof there in a more structural way: we choose $V(t) = \sin(\frac{\pi}{L}t)\, E(t)$ along γ in M and show $I(V,V) < 0$. Here $V(t)$ is the image of a Jacobi field on $\mathbb{S}^n$ via the isomorphism map Φ defined in Lemma 1. □

Remark : ~~From~~ Recall in the proof of Bonnet - Myers Thm, we already show that Cor. 1 (2) can be improved by weakening the sectional curvature restriction to Ricci curvature restriction.

We still miss the generalization of the 2^{nd} part of Sturm comparison theorem: We have not ~~generalized +~~ compared $|\Phi(W)|_{\bar{g}}$ with $|W|_g$ up to the first zero of $\Phi(W)$. ~~for Jacobi~~: Such information is provided by

Theorem 4 (Rauch Comparison Theorem) Let $M, \bar{M}$ be two Rie. manifolds of the same dimension n, and let $\gamma : [a,b] \to M$, $\bar{\gamma} : [a,b] \to \bar{M}$ be normal geodesics, ~~such~~ Let $U, \bar{U}$ be ~~a~~ normal Jacobi fields along $\gamma, \bar{\gamma}$ respectively with ~~the same~~

~~initial data~~: $U(a) = \bar{U}(a) = 0$, and $|\nabla_T U(a)|_g = |\nabla_{\bar{T}} \bar{U}(a)|_{\bar{g}}$.

Suppose :

(1) $\bar{\gamma}$ has no conjugate point on $[a,b]$.

(2) $K(\Pi_{\gamma(t)}) \leq \bar{K}(\bar{\Pi}_{\bar{\gamma}(t)})$ for all $t \in [a,b]$, all 2-dim sections

$\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\bar{\Pi}_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)}\bar{M}$.

Then we have

$$|U(t)|_g \geq |\bar{U}(t)|_{\bar{g}} \quad \text{for all } t \in [a,b].$$

<u>Remark</u>. (a) This is a generalization of the second part of Sturm Comparison Thm. Notice that the Morse-Schoenberg Comparison Thm (Cor. 1) is also a direct consequence of Theorem 4.

(b) " $U, \bar{U}$ be <u>normal</u> Jacobi fields": In fact, we only need require $\underbrace{\langle \dot{U}(a), \dot{\gamma}(a) \rangle = \langle \dot{\bar{U}}(a), \dot{\bar{\gamma}}(a) \rangle = 0}_{(*)}$. This is because, from <u>Prop. 8</u> (p.194), $U = fT + U^{\perp}$ where f is linear. And $f(a) = 0$, $f'(a) = 0$ forces $f \equiv 0$. Hence $(*)$ implies $U = U^{\perp}$.

<u>Proof</u>: If $\bar{U} \equiv 0$, trivial.

If $\bar{U} \not\equiv 0$, then $\bar{U}(t) \neq 0$ for all $t \in (a,b]$, since $\bar{\gamma}$ has no conjugate points.

It suffices to prove that

$$\text{(i)} \lim_{t \to a} \frac{|U(t)|_g}{|\bar{U}(t)|_{\bar{g}}} = 1 \quad \text{and (ii)} \frac{d}{dt} \frac{|U(t)|_g}{|\bar{U}(t)|_{\bar{g}}} \geq 0 \text{ for } t \in (a,b]$$

It turns out it is ~~easier~~ equivalent (but much easier) to consider the norm square.

$$\text{(i)}' \lim_{t \to a} \frac{\langle U(t), U(t) \rangle_g}{\langle \bar{U}(t), \bar{U}(t) \rangle_{\bar{g}}} \text{ and (ii)}' \frac{d}{dt} \frac{\langle U(t), U(t) \rangle_g}{\langle \bar{U}(t), \bar{U}(t) \rangle_{\bar{g}}} \geq 0 \text{ for } t \in (a,b].$$

To prove ~~for~~ (i), we note that:

$$\lim_{t\to a}\frac{\langle U(t),U(t)\rangle_g}{\langle \bar U(t),\bar U(t)\rangle_{\bar g}} \overset{\text{L'Hôpital's Rule}}{=} \lim_{t\to a}\frac{\langle U(t),\nabla_T U(t)\rangle_g}{\langle \bar U(t),\nabla_{\bar T}\bar U(t)\rangle_{\bar g}}$$

$$\overset{\text{L'Hôpital Rule}}{=} \lim_{t\to a}\frac{\langle \nabla_T U(t),\nabla_T U(t)\rangle_g+\langle U(t),\nabla_T\nabla_T U(t)\rangle_g}{\langle \nabla_{\bar T}\bar U(t),\nabla_{\bar T}\bar U(t)\rangle_{\bar g}+\langle \bar U(t),\nabla_{\bar T}\nabla_{\bar T}\bar U(t)\rangle_{\bar g}}$$

$$\overset{|\nabla_T U(a)|_g=|\nabla_{\bar T}\bar U(a)|_{\bar g}}{=} 1$$

To prove (ii), we note that

$$\frac{d}{dt}\frac{\langle U(t),U(t)\rangle_g}{\langle \bar U(t),\bar U(t)\rangle_{\bar g}}=\frac{2\langle U(t),\nabla_T U(t)\rangle\langle \bar U(t),\bar U(t)\rangle-2\langle U(t),U(t)\rangle\langle \bar U(t),\nabla_{\bar T}\bar U(t)\rangle}{\langle \bar U(t),\bar U(t)\rangle^2}$$

Hence (ii) $\Longleftrightarrow$

$$\langle \bar U(t),\bar U(t)\rangle\langle U(t),\nabla_T U(t)\rangle\ge\langle U(t),U(t)\rangle\langle \bar U(t),\nabla_{\bar T}\bar U(t)\rangle.$$

So for each $t_0\in[a,b]$, it suffices to show that

$$\cancel{\langle U(t_0),\nabla_T U(t_0)\rangle}\ \langle U,\nabla_T U\rangle(t_0)\ge\underbrace{\frac{\langle U,U\rangle(t_0)}{\langle \bar U,\bar U\rangle(t_0)}}_{\text{ii}\ c^2}\langle \bar U,\nabla_{\bar T}\bar U\rangle(t_0)$$

(Recall ~~f~~ from (p.198), for any piecewise C^∞ ~~vector~~ field W along γ, the index form

$$I(W,W)=\int_a^b\{\langle\nabla_T W,\nabla_T W\rangle-\langle R(W,T)T,W\rangle\}\,dt$$

$$=-\int_a^b\langle\nabla_T\nabla_T W+R(W,T)T,W\rangle\,dt+\langle\nabla_T W,W\rangle\Big|_a^b$$

$$-\sum_{j=1}^{k}\langle\nabla_{T(t_j^+)}W-\nabla_{T(t_j^-)}W,\ W\rangle.$$
)

Since $U,\bar U$ are Jacobi fields, with $U(a)=\bar U(a)=0$, we have

$$\langle U,\nabla_T U\rangle_g(t_0)=I_a^{t_0}(U,U)$$

$$\langle \bar U,\nabla_{\bar T}\bar U\rangle_{\bar g}(t_0)=\bar I_a^{t_0}(\bar U,\bar U).$$

So, it remains to show $I_a^{t_0}(U,U)\ge c^2\,\bar I_a^{t_0}(\bar U,\bar U)$.

or $I_a^{t_0}\left(\frac{U}{|U(t_0)|},\frac{U}{|U(t_0)|}\right)\ge\bar I_a^{t_0}\left(\frac{\bar U}{|\bar U(t_0)|},\frac{\bar U}{|\bar U(t_0)|}\right)$

This is possible since $U(t_0)\neq 0$.
$I_a^{t_0}(U,U)\ge\bar I_a^{t_0}(\Phi(W),\Phi(W))>0$
Hence $U(t_0)\neq 0$.

Consider the map Φ constructed in Lemma 1, as the input,

at t_0, we choose ϕ_{t_0} s.t. $\phi_{t_0}\left(\frac{U(t_0)}{|U(t_0)|}\right) = \frac{c\,\bar{U}(t_0)}{|\bar{U}(t_0)|}$. This is possible since $\left|\frac{U}{|U(t_0)|}(t_0)\right| = \left|\frac{\bar{U}}{|\bar{U}(t_0)|}(t_0)\right|$

Then $\Phi\left(\frac{U}{|U(t_0)|}\right)$ is a smooth vector field along $\bar{\gamma}$, s.t.

$$\Phi\left(\frac{U}{|U(t_0)|}\right)(t_0) = \frac{c\,\bar{U}(t_0)}{|\bar{U}(t_0)|}.$$

As in the proof of Theorem 3, we see

$$I_a^{t_0}\left(\frac{U}{|U(t_0)|}, \frac{U}{|U(t_0)|}\right) \geq \bar{I}_a^{t_0}\left(\Phi\left(\frac{U}{|U(t_0)|}\right), \Phi\left(\frac{U}{|U(t_0)|}\right)\right) \qquad (1)$$

Now using the minimizing property of Jacobi field (Lemma 2, p. 206), we have

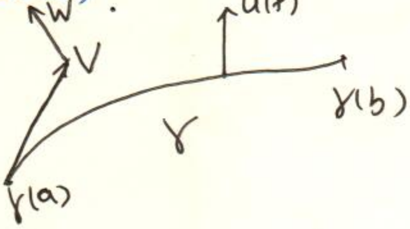
$$\bar{I}_a^{t_0}\left(\Phi\left(\frac{U}{|U(t_0)|}\right), \Phi\left(\frac{U}{|U(t_0)|}\right)\right) \geq \bar{I}_a^{t_0}\left(\frac{c\,\bar{U}}{|\bar{U}(t_0)|}, \frac{c\,\bar{U}}{|\bar{U}(t_0)|}\right) \qquad (2)$$

(This is applicable since $\Phi\left(\frac{U}{|U(t_0)|}\right)(a) = \frac{U(a)}{|U(t_0)|} = 0$, and $\Phi\left(\frac{U}{|U(t_0)|}\right)(t_0) = \frac{c\,\bar{U}(t_0)}{|\bar{U}(t_0)|}$.)

Combining (1) and (2) yields

$$I_a^{t_0}\left(\frac{U}{|U(t_0)|}, \frac{U}{|U(t_0)|}\right) \geq \bar{I}_a^{t_0}\left(\frac{c\,\bar{U}}{|\bar{U}(t_0)|}, \frac{c\,\bar{U}}{|\bar{U}(t_0)|}\right) = c^2\, \bar{I}_a^{t_0}(\bar{U}, \bar{U}) \Rightarrow I_a^{t_0}(U,U) \geq c^2\, \bar{I}_a^{t_0}(\bar{U},\bar{U}). \quad \square$$

Recall from the proof of uniqueness of ~~con~~ simply-connected space forms (Thm 9, p. 217), we have used the idea of comparing the norm of Jacobi field. Theref, we have the <u>same</u> sectional curvatures, and the corresponding Jacobi field has the same norm. (It definitely deserves to read through that proof again with this new perspective.)



Recall for given $t \in [a,b]$, the Jacobi field $U(t)$ at t can be expressed as $U(t) = \left(d\exp_{\gamma(a)}\right)_{(V)}(W)$ for some W.

Hence we have the following equivalent form of Rauch Comparison Theorem.

<u>Theorem 4'</u>. Let $M, \bar{M}$ be two Rie. mflds of the same dimension n.
Let $p \in M, \bar{p} \in \bar{M}$, $\phi : T_pM \to T_{\bar{p}}\bar{M}$ be an ~~vect~~ isometry (of inner product spaces), $V \in T_pM$, $\bar{V} = \phi(V)$.

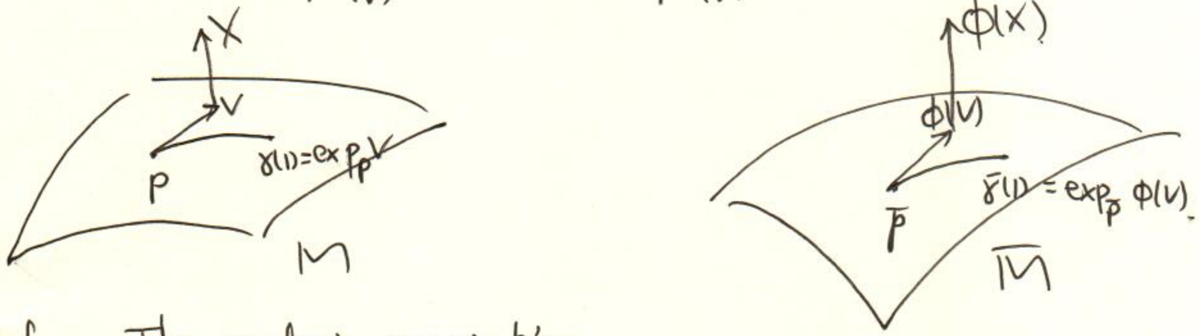
Let $\gamma(t) = \exp_p tV$, $t \in [0,1]$, $\bar{\gamma}(t) = \exp_p t\bar{V}$, $t \in [0,1]$ be geodesic in M, $\bar{M}$ respectively. Let $X \in T_V(T_pM)$, $\phi(X) \in T_{\bar{V}}(T_{\bar{p}}\bar{M})$.

Suppose: (1) $\bar{\gamma}$ has no conjugate point.

(2) $K(\Pi_{\gamma(t)}) \leq \bar{K}(\bar{\Pi}_{\bar{\gamma}(t)})$ for all $t \in [0,1]$, all sections $\Pi_{\gamma(t)} \subset T_{\gamma(t)}M$, $\bar{\Pi}_{\bar{\gamma}(t)} \subset T_{\bar{\gamma}(t)}\bar{M}$.

Then we have

$$(d\exp_p)_{(V)}(X) \geq (d\exp_{\bar{p}})_{(\bar{V})}(\bar{X}).$$



<u>Proof</u>: The geodesic variation

$$F(t,s) = \exp_p t(V + sX)$$

has variational field $U(t)$ which is a Jacobi field s.t.

$$U(0) = 0, \quad \dot{U}(0) = X, \quad U(1) = (d\exp_p)_{(V)}(X).$$

Similarly $\bar{F}(t,s) = \exp_{\bar{p}} t(\bar{V} + s\bar{X})$. gives Jacobi field $\bar{U}(t)$ s.t. $\bar{U}(0) = 0$, $\dot{\bar{U}}(0) = \underset{\bar{X}}{\underset{\parallel}{\phi(X)}}$, $\bar{U}(1) = (d\exp_p)_{(\bar{V})}(\bar{X})$.

Recall from Gauss lemma (Excercise 12, 2),

$$\langle X, V \rangle = \langle (d\exp_p)_{(V)}(V), (d\exp_p)_{(V)}(X) \rangle.$$

and $d\exp_p$ is an isometry along the radial direction, we only need to consider the case $\langle X, V \rangle = 0$ (and, hence, $\langle \phi(X), \phi(V) \rangle = 0$).

therefore, Thm4' follows from Thm 4 (using the Remark (b) on p. 246.)

A particular interesting case :

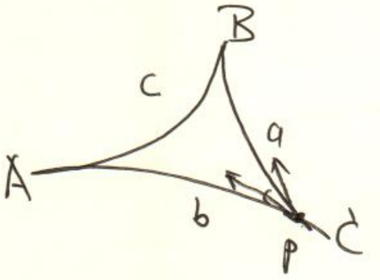
<u>Corollary 2</u> : Let (M,g) be a complete Rie. mfld with nonpositive sectional curvature. Then $\forall\, p \in M$, $\exp_p : T_pM \to M$ satisfies :

$$\left|(d\exp_p)_{(V)}(X)\right| \geq |X| \leftarrow \text{norm of the flat metric on } T_pM.$$

$\forall\, V \in T_pM$, $\forall\, X \in T_V(T_pM) \cong T_pM$. In particular, for any curve $\gamma \subset T_pM$, one has $L(\gamma) \leq L(\exp_p \circ \gamma)$.

<u>Remark</u> : This ~~strengthen~~ strengthen the result Prop. 9 (p. 209) where we show $\exp_p$ has no critical point.

<u>Corollary 3</u> : Let (M,g) be complete <u>simply-connected</u> Riemannian manifold with non-positive sectional curvature. Consider a geodesic triangle in M. (i.e, each side of the triangle is a minimizing geodesic). Let the side lengths are a, b, c with opposite angles A, B, C respectively.

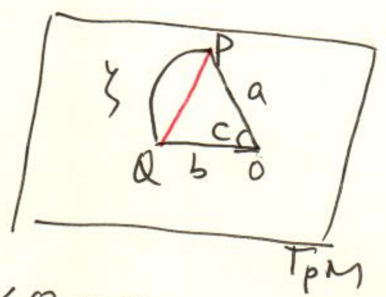


Then

(1) $a^2+b^2-2ab\cos C \leq c$

(2) $A+B+C \leq \pi$.

Moreover, if M has negative sectional curvature, the the inequalities are strict.



<u>Proof</u> : Denote the vertex at the ang C by p.

(1) In T_pM, draw a triangle ΔOPQ, where O is the origin, so. $|OP| = a$, $|OQ| = b$, $\angle O = C$.

In particular, $\exp_p \overrightarrow{OP}$, $\exp_p \overrightarrow{OQ}$ is the other two vertices of the geodesic triangle.

Let ζ be the preimage of the geodesic c in T_pM. Then

$$|PQ| \leq L(\zeta) \underset{\text{Coro. 2}}{\leq} c$$

The Euclidean cosine law tells $|PQ|^2 = a^2+b^2-2ab\cos C$.

(2). Since a, b, c satisfy triangle ineq, we can construct a triangle in $\mathbb{R}^2$ with side lengths a, b, and c. Denote the corresponding opposite angles by A', B', and C'. Then

$$(1) \Rightarrow \quad c^2 \geq a^2+b^2-2ab\cos C \quad \Rightarrow C' \geq C.$$
$$c^2 = a^2+b^2-2ab\cos C'$$

Similarly, we show $A' \geq A$, $B' \geq B$.

Hence $\pi = A'+B'+C' \geq A+B+C$.

When $\sec < 0$. The ineq. in Rauch's thm is also strict. (Check the proof there again). And hence ineq. in Cor 2 is strict. The last conclusion of Corollary 3. then follows. □

Final Remark: "(1) $\bar{\gamma}$ has no conjugate point" in Rauch Comparison theorem is necessary. For example, let us consider two spheres

$$M = S^2(2), \quad \bar{M} = S^2(3).$$

of radius 2, 3 respectively. Let

$$\gamma : [0, 3\pi] \longrightarrow S^2(2), \quad \bar{\gamma} : [0, 3\pi] \longrightarrow S^2(3)$$

(3π is conjugate to 0)

are normal geodesics. Let $W, \bar{W}$ be unit parallel normal vector fields along $\gamma, \bar{\gamma}$ respectively. Then

$$U(t) = 2\sin\frac{t}{2}\, W(t), \quad \bar{U}(t) = 3\sin\frac{t}{3}\, \bar{W}(t)$$

are Jacobi fields s.t. $U(0) = \bar{U}(0) = 0$, $|\dot{U}(0)| = |\dot{\bar{U}}(0)| = 1$.

~~But~~ Recall $\sec(S^2(r)) = \frac{1}{r^2}$. $\sec(S^2(2)) = \frac{1}{4} > \frac{1}{9} = \overline{\sec}(S^2(3))$

But $|\bar{U}(3\pi)| = 0 < |U(3\pi)| = |2\sin\frac{3\pi}{2}| = 2$. ~~Note~~