$$\begin{cases} f''(t) + k f = 0 \\ f(0) = 0,\ f(b) = 1 \end{cases}$$

By the construction, we know $\Phi(U_i) = f(t) e_i(t)$.

Any normal Jacobi field $\bar{U}(t)$ along $\bar{\gamma}$ with $\bar{U}(0) = 0$ can be ~~written~~ expressed as a linear combination of $\Phi(U_i)$, $i = 2, \ldots, n$.

$$\bar{U}(t) = \sum_{i=2}^{n} c_i f(t) e_i(t), \quad c_i \in \mathbb{R}.$$

$$= f(t) \underbrace{\sum_{i=2}^{n} c_i e_i(t)}_{\to \text{ parallel}}.$$

By Jacobi equation, $\underbrace{\Phi(U_i)}_{= f(t) e_i(t)}$ is a Jacobi field implies

$$\langle R(e_i, T) T, e_i \rangle (t) = k. \qquad \square$$

<u>Corollary 8</u>. Under the same assumption of Thm 9, and let $f : [a, b] \to \mathbb{R}$ be a smooth function with $f' \geq 0$. Then

$$\Delta f(\rho)(\gamma(t)) \geq \bar{\Delta} f(\bar{\rho})(\bar{\gamma}(t)), \quad \forall t \in [a, b].$$

<u>Proof</u>: $\Delta f(\rho) = f''(\rho) + f'(\rho) \Delta \rho$.

$$\bar{\Delta} f(\bar{\rho}) = f''(\bar{\rho}) + f'(\bar{\rho}) \bar{\Delta} \bar{\rho} \qquad \square$$

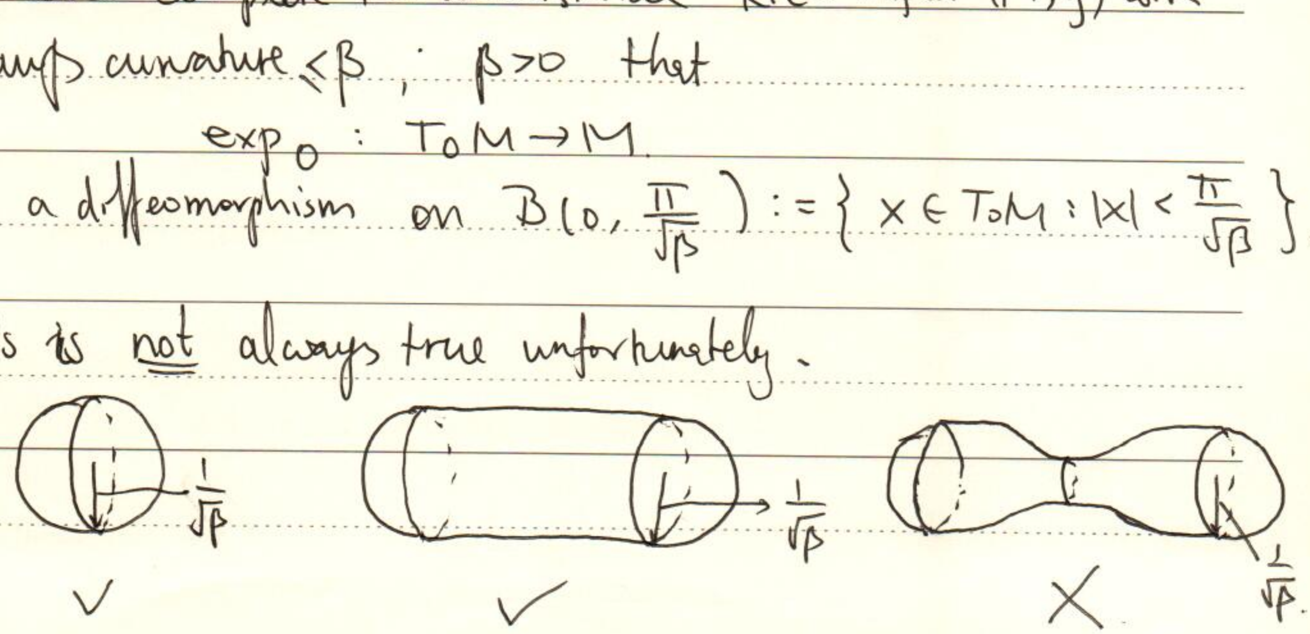
## §5½. <u>Comments on injectivity radius estimate and sphere theorems</u>:

Excercise 13, ~~we~~ ~~was~~ asked to show for a simply-connected complete two dimensional Rie-mfld $(M, g)$ with Gauss curvature $\leq \beta$, $\beta > 0$ that

$$\exp_0 : T_0 M \to M$$

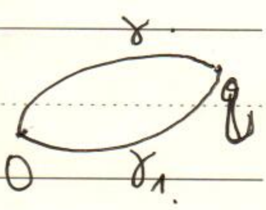
is a diffeomorphism on $B(0, \frac{\pi}{\sqrt{\beta}}) := \{ x \in T_0 M : |x| < \frac{\pi}{\sqrt{\beta}} \}$.

This is <u>not</u> always true unfortunately.

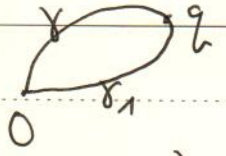


By comparison theorem, Gauss curvature $\leq \beta$ implies
$$\exp_O(B(0,\tfrac{\pi}{\sqrt{\beta}})) \text{ contains no conjugate point of } O.$$
But this is not enough to conclude that $\exp_O$ is a diffeomorphism.
Actually, in other words, we are asked to show
$$inj_M \geq \frac{\pi}{\sqrt{\beta}}.$$
For that ~~pa~~ result, we need further restrict Gauss curvature $>0$.
This is actually Klingenberg's injectivity radius estimate.
Theorem 10 (Klingenberg, 1959) Suppose $(M,g)$ is an orientable even-dimensional manifold with $0 <$ sectional curvature $\leq \beta$. Then $inj_M \geq \frac{\pi}{\sqrt{\beta}}$. If $M$ is not orientable, then $inj_M \geq \frac{\pi}{2\sqrt{\beta}}$.

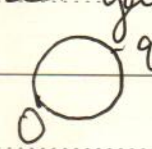
For the proof, we refer to [Spivak IV, 34-36], or [PP, Chap 8: §6.2]
Here, we explain the rough ideas:
We need show $\exp_O(B(0,\frac{\pi}{\sqrt{\beta}}))$ has no cut point. If there is a cut point $q$ of $O$ along $\gamma$; since $q$ cannot be a conjugate point of $O$, ~~By~~ we have by Theorem 5 (p.255), there exists exactly two minimal geodesics from $O$ to $q$.



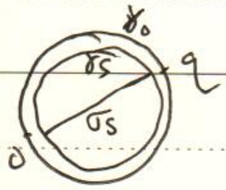

In fact one can further argue that when $q$ is the closest one to $O$ in $C(O)$, $\dot\gamma|_q = -\dot\gamma_1|_q$.
Moreover, when $O$ is a point such that ~~O~~ it is the minimum point of the function $d(p, C(p))$, we have the two ~~geodesics~~ geodesics $\gamma, \gamma_1$ give a closed geodesic.




Recall in the proof of Synge theorem, under the assumption "orientable, even-dim'l".



any closed geodesic has a variation $F(t,s)$
such that the curves $\gamma_s(t) := F(t,s)$ has length
$$L(\gamma_s(t)) < L(\gamma_0(t)), \quad \text{when } 0<s \text{ small.}$$
Hence the whole curve $\gamma_s$ lie in the interior of the cut locus. So there exists a minimal geodesic $\sigma_s$ from $\gamma_s(0)$ to the farthest point $\gamma_s(t_s)$ along $\gamma_s$.




Choose subsequence if necessary, those geodesics $\sigma_s$ converge to a minimal geodesic $\sigma$ from $o$ to $q$ which is different from $\gamma$, and $\gamma_1$. This contradicts to the assumption that $q$ is a cut point and $\gamma, \gamma_1$ are the only minimizing geodesic from $O$ to $q$. So there is no cut point in $\exp_o\left(B(o, \frac{\pi}{\sqrt{\beta}})\right)$. $\square$

A much deeper result by Klingenberg asserts that if a simply-connected manifold has all its sectional curvatures in the interval $(\frac{1}{4}\beta, \beta]$, then $\mathrm{inj}_M \geq \frac{\pi}{\sqrt{\beta}}$. There are further improvement on the left-end of the interval.

Actually, those injectivity radius estimate and Rauch comparison theorem are crucial tools to establish fascinating Sphere Theorems:

Theorem 11. (Topological Sphere Theorem). Let $(M,g)$ be a simply-connected complete Riemannian manifold. Suppose $M$ has all its sectional curvatures in the interval $(\frac{1}{4}\beta, \beta]$, $\beta>0$. then $M$ is homeomorphic to the sphere.

This result is due to Rauch (prove in the case $\sec \in (\frac{3}{4}\beta, \beta]$), ~~and~~ Klingenberg, Berger.

<u>Very brief explanation</u>: In topology, <u>Brown theorem</u> tells that: if a compact manifold $M$ is the union of two open sets, each of which is diffeomorphic to $\mathbb{R}^n$, then $M$ is diffeomorphic to $\mathbb{S}^n$.

So let $p, q \in M$, s.t. $d(p,q) = \operatorname{diam}(M,g) \underset{\Uparrow}{\overset{\sec \geq \frac{1}{4}\beta}{\leq}} \frac{2\pi}{\sqrt{\beta}}$

$$\exp_p : T_pM \to M, \quad \exp_q : T_qM \to M$$

are diffeomorphisms on $B(0,\delta_p)$, $B(0,\delta_q)$ at least when $\delta_p$, $\delta_q$ are small enough.

$$M \supset \exp_p(B(0,\delta_p)) \cup \exp_q(B(0,\delta_q)).$$

On the other hand, if we have $\delta_p, \delta_q$ large enough, we have

$$M = \exp_p(B(0,\delta_p)) \cup \underset{\uparrow \text{ hom. to } \mathbb{R}^n}{\exp_q(B(0,\delta_q))}.$$

~~Rink~~ Scaling $\beta$ to be 1. $\operatorname{diam}(M,g) \leq 2\pi$.

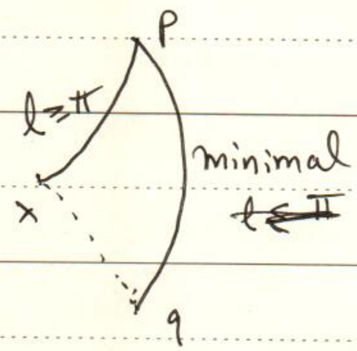
Klingenberg $\Rightarrow$ $\operatorname{inj}_M \geq \pi$.

It remains to show

$$M \subset \exp_p(B(0, \operatorname{inj}(p))) \cup \exp_q(B(0, \operatorname{inj}(q))).$$

That is for any $x \in M$, if $d(p,x) \geq \operatorname{inj}_M \geq \pi$, then we need show $d(q,x) < \operatorname{inj}_M$.

For this purpose, we need a global version of the Rauch theorem: Toponogov triangle comparison theorem.
(This has been discussed in the tutorial).



## §6. Volume Comparison Theorems.

Now let us come back to investigate ~~anoth~~ a geometric quantity which we have discussed ~~for since~~ at the very beginning of this course: the volume.

Recall $E(p) := \{tV : V \in S_p \text{ and } 0 \le t < \tau(V)\}$ from p. 260.
Let us denote by $E_p := \exp_p E(p)$. We have shown that

$$\exp_p : E(p) \to E_p$$

is a diffeomorphism, and $E(p)$ is diffeomorphic to an open ball. Therefore $E_p$ is also diffeomorphic to an open ball. Since the cut locus is of zero measure, we have

$$Vol(M) = \int_M dvol = \int_{E_p} dvol$$

Note $E_p \subset M$ can be considered as a coordinate neighborhood!

Hence $Vol(M) = \int_{E_p} \sqrt{\det(g_{ij})}\, \underbrace{dx^1 \cdots dx^n}_{\text{Lebesgue measure}}$

$\downarrow$

$\sqrt{\det(g_{ij})\circ x^{-1}}$ more precisely.

~~we can always~~ $= \int_{E_p} \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$.

(This is another way to see the def. of volume does not depend on orientability).

For the ball $B_p(r) := \{ q \in M \mid d(p,q) < r\}$ we have

$$vol(B_p(r)) = \int_{B_p(r)} dvol = \int_{B_p(r) \cap E_p} \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

<u>How to calculate $vol(B_p(r))$ in a Rie. manifold?</u>

We first prepare some algebraic results: Let $f : \tilde{V} \to V$ be a linear transformation between two $n$-dim'l inner product spaces. Let $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, $\{e_1, \cdots, e_n\}$ be their orthonormal basis, and $\{\tilde{\omega}^1, \cdots, \tilde{\omega}^n\}$, $\{\omega^1, \cdots, \omega^n\}$ be their dual basis.

Let $\tilde{\Omega} = \tilde{\omega}^1 \wedge \cdots \wedge \tilde{\omega}^n$, $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$.

Then $f^*\Omega$ is defined as

$$f^*\Omega(\tilde{X}_1, \cdots, \tilde{X}_n) := \Omega(f(\tilde{X}_1), \cdots, f(\tilde{X}_n)).$$
$$\tilde{X}_i \in \tilde{V}$$

Therefore, $\exists\ a_0 \in \mathbb{R}$ s.t $f^*\Omega = a_0 \tilde{\Omega}$.

On the other hand, Let

$$f(\tilde{e}_i) = \sum_{j=1}^{n} \alpha_i^j e_j, \quad i = 1, 2, \cdots, n.$$

Then $a_0 = \det[\alpha_i^j] =: \det(f)$.

<u>Claim</u> Let $\{\tilde{A}_1, \cdots, \tilde{A}_n\}$ be a basis of $\tilde{V}$. then

$$|a_0| = |\det(f)| = \left|\frac{f^*\Omega}{\tilde{\Omega}}\right| = \frac{|f(\tilde{A}_1) \wedge \cdots \wedge f(\tilde{A}_n)|}{|\tilde{A}_1 \wedge \cdots \wedge \tilde{A}_n|}.$$

<u>Proof</u>: Recall $\forall X_i \in V$, $i = 1, \cdots, n$, we have

$$|X_1 \wedge \cdots \wedge X_n| = \langle X_1 \wedge \cdots \wedge X_n, X_1 \wedge \cdots \wedge X_n \rangle^{1/2}$$
$$= \sqrt{\det[\langle X_i, X_j \rangle]}.$$

In particular, if $\{X_i\}$ are orthonormal, $|X_1 \wedge \cdots \wedge X_n| = 1$.

Therefore, letting $\tilde{A}_i = \sum_{j=1}^{n} \beta_i^j \tilde{e}_j$, $i = 1, 2, \ldots, n$.

we have

$$|f(\tilde{A}_1) \wedge \cdots \wedge f(\tilde{A}_n)| = |\det[\beta_i^j]| \cdot |f(\tilde{e}_1) \wedge \cdots \wedge f(\tilde{e}_j)|$$
$$= |\det[\beta_i^j]| \cdot |\det[\alpha_i^j]|\ |e_1 \wedge \cdots \wedge e_n|$$
$$= |\det[\beta_i^j]| \cdot |\det(f)|.$$

On the other hand,

$$|\tilde{A}_1 \wedge \cdots \wedge \tilde{A}_n| = |\det[\beta_i^j]|\ |\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n| = |\det[\beta_i^j]|.$$

This proves the claim $\square$

Now, let $\gamma : [0, b] \to M$ be a normal geodesic with $\gamma(0) = p$.

Let $\tilde{\Omega}$ be ~~the~~ a volume $n$-form of the Euclidean space $T_pM$, and $\Omega(t)$ be a volume $n$-form at $\gamma(t) \in M$. Then we have

$$\exp_p : T_pM \to M$$

$$(d\exp_p)_{t\dot\gamma(0)} : T_{t\dot\gamma(0)}(T_pM) \to T_{\gamma(t)}M.$$

Define a function $\varphi : [0,b] \to \mathbb{R}$ to be

$$\varphi(t) := \left| \frac{(d\exp_p)^*_{(t\dot\gamma(0))}(\Omega(t))}{\tilde\Omega} \right|.$$

Note that $\Omega(t)$, $\tilde\Omega$ depend on the choices of different orientations, but $\varphi(t)$ does not. We can always choose proper orientation s.t.

$$\varphi(t) = \frac{(d\exp_p)^*(\Omega(t))}{\tilde\Omega}$$ (we omit the subscript: $(t\dot\gamma(0))$)

(Because we are working in the single coordinate neighborhood $E_p$ !!)

We in fact can define a function $\varphi : E(p) \to \mathbb{R}$ as below:

$\forall\, \tilde y \in E(p)$, $$\varphi(\tilde y) := \left| \frac{(d\exp_p)^*(\Omega(y))}{\tilde\Omega(\tilde y)} \right| = \frac{(d\exp_p)^*(\Omega(y))}{\tilde\Omega(\tilde y)}.$$

where $y = \exp_p \tilde y$

Then we can rewrite the volume of $M$ as

$$Vol(M) = vol(E_p) = \int_{E_p} \Omega = \int_{\exp_p(E(p))} \Omega$$

$$= \int_{E(p)} (d\exp_p)^*\Omega = \int_{E(p)} \varphi\,\tilde\Omega = \int_{E(p)} \varphi\, d\,vol_{T_pM}$$

$\uparrow$ standard Lebesgue measure

and $$Vol(B_p(r)) = \int_{B(0,r)\cap E(p)} \varphi\, dvol_{T_pM}.$$

So the keypoint is to compute the function $\varphi$, for which we need employ results about Jacobi fields again.

Lemma 26. Let $\gamma: [0,b] \to M$ be a normal geodesic containing no conjugate point. Let $J_1, \cdots, J_{n-1}$ be $(n-1)$ linearly independent normal Jacobi fields along $\gamma$ with $J_i(0)=0$, $i=1,2,\cdots,n-1$. Then we have

$$\varphi(t) = \frac{|J_1(t)\wedge\cdots\wedge J_{n-1}(t)|}{t^{n-1}|\dot J_1(0)\wedge\cdots\wedge \dot J_{n-1}(0)|}, \quad t\in(0,b].$$

Proof. $d\exp_p : T_{\gamma(t)}T_pM \to T_{\gamma(t)}M$.

Note that $J_1(t),\cdots,J_{n-1}(t), \dot\gamma(t)$ are $n$ linearly independent vectors in $T_{\gamma(t)}M$, and hence form a basis of $T_{\gamma(t)}M$. By "dimension consideration", this also implies

$\dot J_1(0),\cdots,\dot J_{n-1}(0)$ are linearly independent.

$\langle J_i(t),\dot\gamma(t)\rangle = 0,\ t\in[0,b] \Rightarrow \langle \dot J_i(0),\dot\gamma(0)\rangle = 0$. Hence

$\dot J_1(0),\cdots,\dot J_{n-1}(0), \dot\gamma(0)$ form a basis of $T_pM$.

Recall for the variation $F(t,s) := \exp_p t(\dot\gamma(0)+sW)$, we have its variational field $U(t)$ ~~satisfies~~ is a Jacobi field with $U(0)=0$, $\dot U(0)=W$, and $U(t) = (d\exp_p)_{(t\dot\gamma(0))}(tW)$.
Pick $W = \dot J_i(0)$, we then obtain

$$J_i(t) = (d\exp_p)_{(t\dot\gamma(0))}(t\dot J_i(0))$$

Notice that for any $t\in(0,b]$,

$\dot\gamma(0), t\dot J_1(0),\cdots, t\dot J_i(0)$ also form a basis of $T_pM$.

$$\text{Hence}\quad \varphi(t) = \frac{|\dot\gamma(t)\wedge J_1(t)\wedge\cdots\wedge J_{n-1}(t)|}{|\dot\gamma(0)\wedge t\dot J_1(0)\wedge\cdots\wedge t\dot J_{n-1}(0)|} \underset{\substack{\text{since } \langle\dot\gamma(t),J_i(t)\rangle=0 \\ |\dot\gamma(t)|=1 \\ \langle\dot\gamma(0),\dot J_i(0)\rangle=0}}{=} \frac{|J_1(t)\wedge\cdots\wedge J_{n-1}(t)|}{t^{n-1}|\dot J_1(0)\wedge\cdots\wedge\dot J_{n-1}(0)|}$$

□

<u>Theorem 12</u>. (Bishop) Let $M$ be a Rie. mfld with

$$Ric \geq (n-1)k \quad ( \cancel{k \geq 0 \text{ or } k \text{ imaginary number}} \; k \in \mathbb{R}.$$

Let $\gamma : [0,b] \to M$ be a normal geodesic containing no cut point then

$$\frac{\varphi(t)}{\varphi_k(t)} \text{ is non increasing. } \quad t \in (0,b]$$

$\varphi_k(t)$: the function on the simply-connected space form $M^k$ of sectional curvature $k$

By Lemma 2, we can check $\varphi_k(t) = \left(\frac{f_k(t)}{t}\right)^{n-1}$ where $f_k(t) = \begin{cases} t & k=0 \\ \frac{1}{\sqrt{k}} \sin \sqrt{k}t, & k>0 \\ \frac{1}{\sqrt{-k}} \sinh \sqrt{-k}t, & k<0. \end{cases}$

We will show this result by reduce it to the Laplacian comparison ~~by~~ via the following Lemma.

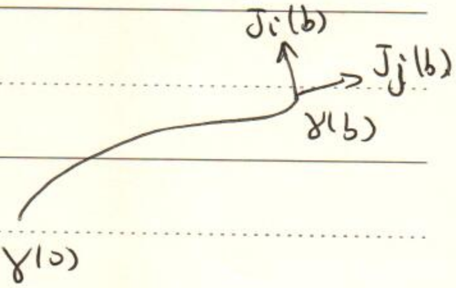
<u>Lemma 3</u>. Let $\gamma : [0,b] \to M$ be a normal geodesic with no cut point of $\gamma(0)$. Let $\rho(x) := d(x, \gamma(0))$.

Then
$$\frac{\varphi'}{\varphi}(t) = \left(\Delta\rho - \frac{n-1}{\rho}\right)(\gamma(t)), \quad t \in (0,b] \qquad (*)$$

<u>Proof</u>: We only need prove $(*)$ at $\gamma(b)$.

Let $J_1, \cdots, J_{n-1}$ be Jacobi fields along $\gamma$ with $J_i(0)=0$, $i=1,2,\cdots,n-1$ s.t.

$$\langle J_i(b), J_j(b) \rangle = \delta_{ij}, \quad 1 \leq i,j \leq n-1.$$




~~Then~~ Recall $\operatorname{Hess}\rho(\dot\gamma, \dot\gamma) = 0$. Hence we have

$$\Delta\rho(\gamma(b)) = \sum_{i=1}^{n-1} \operatorname{Hess}\rho(J_i(b), J_i(b))$$

① 
$$= \sum_{i=1}^{n-1} I(J_i(b), J_i(b)) = \sum_{i=1}^{n-1} \langle \nabla_T \dot{J}_i(b), J_i(b) \rangle$$

On the other hand,

$$\frac{\varphi'}{\varphi}(b) = \frac{\frac{d}{dt}\varphi^2}{2\varphi^2}(b) \quad \cancel{\text{where}} \quad \varphi(b) = \frac{|J_1(b) \wedge \cdots \wedge J_{n-1}(b)|}{b^{n-1}|\dot{J}_1(0) \wedge \cdots \wedge \dot{J}_{n-1}(0)|}$$

$$= \frac{1}{b^{n-1}|\dot{J}_1(0) \wedge \cdots \wedge \dot{J}_n(0)|}$$

$$\frac{d}{dt}\varphi^2(b) = \frac{d}{dt}\Big|_{t=b} \frac{\langle J_1(t)\wedge\cdots\wedge J_{n-1}(t),\ J_1(t)\wedge\cdots\wedge J_{n-1}(t)\rangle}{t^{2n-2}|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_n(0)|^2}$$

$$= \frac{2\sum_{i=1}^{n-1}\langle J_1(t)\wedge\cdots\wedge\dot{J}_i(t)\wedge\cdots\wedge J_{n-1}(t),\ J_1(t)\wedge\cdots\wedge J_{n-1}(t)\rangle}{t^{2n-2}|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_n(0)|^2}\Bigg|_{t=b}$$

$$- 2(n-1)\frac{|J_1(t)\wedge\cdots\wedge J_{n-1}(t)|^2}{t^{2n-1}|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_{n-1}(0)|^2}\Bigg|_{t=b}$$

$$= 2\sum_{i=1}^{n-1}\det\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \langle\dot{J}_i(b),J_1\rangle & & \cdots & \langle\dot{J}_i(b),J_{n-1}(b)\rangle\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\Bigg/ b^{2n-2}|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_n(0)|^2$$

$$- \frac{2(n-1)}{b^{2n-1}|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_{n-1}(0)|^2}$$

$$= \frac{2}{|\dot{J}_1(0)\wedge\cdots\wedge\dot{J}_{n-1}(0)|^2}\left(\frac{\sum_{i=1}^{n-1}\langle\dot{J}_i(b),J_i(b)\rangle}{b^{2n-2}} - \frac{n-1}{b^{2n-1}}\right)$$

Hence, we calculate

$$②\quad \frac{\varphi'(b)}{\varphi(b)} = \frac{\frac{d}{dt}\varphi^2}{2\varphi^2}(b) = \sum_{i=1}^{n-1}\langle\dot{J}_i(b),J_i(b)\rangle - \frac{n-1}{b}$$

Combining ① and ②, we obtain

$$\frac{\varphi'}{\varphi}(b) = \Delta\rho(\gamma(b)) - \frac{n-1}{\rho(\gamma(b))}$$

$$= \left(\Delta\rho - \frac{n-1}{\rho}\right)(\gamma(b)). \qquad \square$$

<u>Proof of Theorem 12</u>. Let $\bar{\gamma}:[0,b]\to M^{k_0}$ be a normal geodesic in the simply-connected space form $M^{k_0}$ of sectional curvature $\equiv k_0$ ( ~~negative if k is imaginary~~ ). Note that

(✱✱) $\bar{\gamma}$ has no cut point $\Leftrightarrow$ $b < \frac{\pi}{\sqrt{k}}$ ( $\frac{\pi}{\sqrt{k}} = \infty$ if $k<0$ ~~k = imaginary number~~ )

By our assumption $\gamma : [0,b] \to M$ is a normal geodesic in $M$ without cut point, hence $b < \frac{\pi}{\sqrt{k}}$ by Bonnet–Myers Thm.

Therefore, (✱✱) implies $\bar{\gamma} : [0,b] \to M^{k^0}$ has no cut point too.

So the Laplacian comparison Theorem 9 (p. 268) is applicable.

Hence Lemma 3 tells
$$\frac{\varphi'}{\varphi}(t) \geq \frac{\varphi_{k^0}'}{\varphi_{k^0}}(t).$$

Hence $(\ln\varphi)'(t) \geq (\ln \varphi_{k^0})'(t)$

which implies
$$(\ln\varphi - \ln\varphi_{k^0})'(t) = \left(\ln \frac{\varphi}{\varphi_{k^0}}\right)'(t) \geq 0.$$

This means $\left(\frac{\varphi}{\varphi_{k^0}}\right)'(t) \geq 0.$

This completes the proof. $\square$

Theorem 13 (Bishop–Gromov) If $(M,g)$ is a complete Rie. mfld with $Ric \geq (n-1)k^0$, ~~(k ≥ 0 or k imaginary)~~ $k \in \mathbb{R}$.

Let $p \in M$ be an arbitrary point. Then the function
$$r \longmapsto \frac{vol(B_p(r))}{vol(B_{\bar{p}}^{k^0}(r))}$$
is nondecreasing, where $B_{\bar{p}}^{k^0}(r)$ is a geodesic ball of radius $r$ in the simply-connected space form $M^{k^0}$.

Corollary 9. (Bishop). If $(M,g)$ is a complete Riem fld with
$$Ric \geq (n-1)k^0 > 0$$

Then $Vol(M) \leq Vol\left(S^n\left(\frac{1}{\sqrt{k}}\right)\right)$

The equality holds iff $M$ is isometric to $S^n\left(\frac{1}{\sqrt{k}}\right)$.

Proof of Theorem 13. What is $vol(B_p(r))$? First note that when $B_p(r) \subset \bar{E}_p$, we have

$$vol(B_p(r)) = \int_{B_p(r)} dvol_M = \int_{B(0,r)} \varphi \, dvol_{T_pM}$$

↳ standard Lebesgue measure.

change to polar coord.

$$\stackrel{\Downarrow}{=} \int_0^r \int_{S^{n-1}} \varphi(t,\theta) t^{n-1} dt \, d\theta.$$

$S^{n-1}$: unit sphere; $d\theta$: standard volume element on the unit sphere $S^{n-1}$. (Euclidean)

The assumption $B_p(r) \subset E_p$ means $r\theta \in E(p)$ for any $\theta \in S^{n-1}$.

How to go beyond cut point?

Let $\chi$ be the characteristic function of $E(p) \subset T_pM$, i.e.,

$$\chi(r,\theta) = \begin{cases} 1, & \text{when } (r,\theta) \in E(p) \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $B_p(r) \subset M$,

$$vol(B_p(r)) = \int_{B(0,r)\cap E_p} \varphi \, dvol_{T_pM} = \int_{B(0,r)} \chi\varphi \, dvol_{T_pM}$$

$$= \int_0^r \int_{S^{n-1}} \chi(t,\theta) \varphi(t,\theta) t^{n-1} dt \, d\theta.$$

Remark: Recall that for $(t,\theta) \in E(p)$, $\varphi(t,\theta) t^{n-1} = \dfrac{|J_1(t) \wedge \cdots \wedge J_{n-1}(t)|}{|\dot{J}_1(0) \wedge \cdots \wedge \dot{J}_{n-1}(0)|}$.

On the simply-connected space-form $M^k$, we have

$$\varphi_k(t,\theta) t^{n-1} = \varphi_k(t) t^{n-1} = (f_k(t))^{n-1}$$

where

$$f_k(t) = \begin{cases} t, & k=0 \\ \frac{1}{\sqrt{k}} \sin\sqrt{k}\, t, & k>0 \\ \frac{1}{\sqrt{-k}} \sinh\sqrt{-k}\, t, & k<0 \end{cases}.$$

We also define the characteristic function $\chi_k$ on $M^k$. Recall, when $k \le 0$, $\chi_k \equiv 1$, when $k>0$, $\chi_k \doteq 0$ only at the one point,

Bonnet-Myers: $Ric \ge (n-1)k$ $\Rightarrow$ diam $\le \frac{\pi}{\sqrt{k}}$ | since $\forall p \in M^k$, $C(p) = -p$ and $d(p, C(p)) = \frac{\pi}{\sqrt{k}}$.

Therefore ~~what~~ we have $\chi(r,\theta) \leq \chi_k(r,\theta) = \chi_k(r)$ (independent of $\theta$)

That is, the function

$$r \longmapsto \frac{\chi(r,\theta)}{\chi_k(r)} \quad \left(\text{where we use } \frac{0}{0} = 0\right)$$

is nonincreasing.

Recall Theorem 12 tells $r \mapsto \dfrac{\varphi(r,\theta)}{\varphi_k(r)}$ is non-increasing. $(r < \tau(\theta))$

Hence $r \longmapsto \dfrac{\chi(r,\theta)\,\varphi(r,\theta)}{\chi_k(r)\,\varphi_k(r)}$ is non-increasing. $(\#)$

Consider the function $a(t) = \int_{S^{n-1}} \chi\varphi(t,\theta)\, t^{n-1}\, d\theta$.

$$a_k(t) = \int_{S^{n-1}} \chi_k \varphi_k(t)\, t^{n-1}\, d\theta.$$

First observe $a_k(t) = \chi_k \underbrace{\varphi_k(t) t^{n-1}}_{(f_k(t))^{n-1}} \underbrace{vol(S^{n-1})}_{\text{constant}}$.

Hence we have $\dfrac{a(t)}{a_k(t)} = \dfrac{1}{vol(S^{n-1})} \int_{S^{n-1}} \dfrac{\chi\varphi(t,\theta) \cdot \cancel{t^{n-1}}}{\chi_k \varphi_k(t)\ \cancel{t^{n-1}}}\, d\theta$

$(\#)$ then implies ~~imp~~ immediately that

$$t \longmapsto \frac{a(t)}{a_k(t)} \text{ is non increasing.}$$

This tells $r \longmapsto \dfrac{Vol(B_p(r))}{Vol(B_o^k(r))} = \dfrac{\int_0^r a(t)\,dt}{\int_0^r a_k(t)\,dt}$ is nonincreasing

due to the following Lemma 4. $\square$

Lemma 4. Let $f, g : [0,\infty) \to (0,\infty)$ be two positive functions and the function $t \longmapsto \dfrac{f(t)}{g(t)}$ is non increasing.

Then the function $t \longmapsto \dfrac{\int_0^t f}{\int_0^t g}$ is also nonincreasing.

Proof: Let us denote $h = \frac{f}{g}$. For $t_1 < t_2$, we ~~have~~ hope to show $\frac{\int_0^{t_1} f}{\int_0^{t_1} g} \geq \frac{\int_0^{t_2} f}{\int_0^{t_2} g}$.

i.e. $\int_0^{t_1} f \cdot \int_0^{t_2} g \geq \int_0^{t_2} f \int_0^{t_1} g$.

We observe
$$\int_0^{t_1} f \cdot \int_0^{t_2} g = \int_0^{t_1} f \cdot \int_0^{t_1} g + \int_0^{t_1} f \int_{t_1}^{t_2} g$$
$$\int_0^{t_2} f \int_0^{t_1} g = \int_{t_1}^{t_2} f \int_0^{t_1} g + \int_0^{t_1} f \int_0^{t_1} g.$$

Hence it remains to show $\int_0^{t_1} f \int_{t_1}^{t_2} g \geq \int_{t_1}^{t_2} f \int_0^{t_1} g$.

This follows from the calculation:
$$\int_0^{t_1} f \int_{t_1}^{t_2} g = \int_0^{t_1} gh \int_{t_1}^{t_2} g \geq \left(\int_0^{t_1} g\right) h(t_1) \int_{t_1}^{t_2} g$$
$$\geq \left(\int_0^{t_1} g\right)\left(\int_{t_1}^{t_2} hg\right) = \int_0^{t_1} g \int_{t_1}^{t_2} f. \quad \square$$

Remark: Recall we actually have
$$\varphi(r,\theta) = \sqrt{\det(g_{ij})\circ x^{-1}}\,(r,\theta) \quad r < \tau(\theta)$$
$$\varphi_k(r) = \left(\frac{f_k(r)}{r}\right)^{n-1}.$$
$$\lim_{r\to 0} \varphi(r,\theta) = 1. \Rightarrow \lim_{r\to 0} \frac{\varphi(r,\theta)}{\varphi_k(r)} = 1.$$

Hence Theorem 12 $\Rightarrow$ $\varphi(r,\theta) \leq \varphi_k(r)$. when $r < \tau(\theta)$.

$$\Rightarrow \quad \chi\varphi(r,\theta) \leq \chi_k \varphi_k(r)$$
$$\Rightarrow \quad \underset{\mathrm{vol}(B_p(\frac{\pi}{\sqrt{k}}))}{\underset{\shortparallel}{\mathrm{vol}(M)}} \leq \mathrm{vol}(M_k) \qquad (**)$$

In fact, one have $\lim_{r\to 0} \frac{\mathrm{Vol}(B_p(r))}{\mathrm{Vol}(B^k(r))} = 1$. So $(**)$ can also be derived from Thm 13.

Proof of Corollary 9 : Recall the sphere of radius $\frac{1}{\sqrt{k}}$ has constant sectional curvature $k$. Hence Thm 12 $\Rightarrow$

$$Vol(M) \le Vol(\mathbb{S}^n(\tfrac{1}{\sqrt{k}}))$$

If "=" holds, then all inequalities in the proof of Thm 12 should be "=". Particularly,

$$\Delta \rho(\gamma(t)) = \Delta \rho_k(\bar{\gamma}(t)) \quad \text{For any } t \text{ s.t. } t\dot{\gamma}(0) \in E(\gamma(0)) \subset T_{\gamma(0)}M$$
$$\forall \gamma.$$

Recall ~~by~~ from the Laplacian comparison theorem 9 (p.268), this means any section in $T_{\gamma(t)}M$ containing $\dot{\gamma}(t)$ has sectional curvature $k$. Since $\gamma$ is arbitrary, we have $M$ has constant sectional curvature $k$. So its universal covering space is isometric to $\mathbb{S}^n(\frac{1}{\sqrt{k}})$ (by the uniqueness of simply-connected space-forms). But since $Vol(M) = Vol(\mathbb{S}^n(\frac{1}{\sqrt{k}}))$, we have $M$ is isometric to $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. □

Applications of Volume Comparison :

(i) $\boxed{\text{Maximal diameter Theorem}}$

Theorem 14 (郑绍远 Maximal diameter Theorem 1975). Let $M$ be a complete Riemannian manifold with

$$Ric \ge (n-1)k > 0 \quad \text{and} \quad diam_M = \frac{\pi}{\sqrt{k}}.$$

Then $M$ is isometric to $\mathbb{S}^n(\frac{1}{\sqrt{k}})$. ($\mathbb{S}^n(\frac{1}{\sqrt{k}})$ has const. sectional curvature $k$)

Remark (1) This is a good complement of Bonnet - Myers Diameter Estimate: the "=" holds in Bonnet - Myers iff $M$ is isometric to $\mathbb{S}^n(\frac{1}{\sqrt{k}})$.

(2). When assuming $sec \ge k > 0$, this result has been proved by Toponogov in 1959. 郑's original proof uses his comparison theorems for first eigenvalues. Shiohama (Trans. AMS. 1983) gives a much more elementary proof using the Volume Comparison.

Proof of Theorem 14: By scaling, we only need deal with the case $k=1$.

Let $p, q \in M$ be two points such that $d(p,q) = \mathrm{diam}_M = \pi$.

Then $\underbrace{B_p(r) \cap B_q(\pi - r)}_{\text{open balls}} = \phi, \ \forall r \in [0,\pi]$.

Hence $\mathrm{Vol}(B_p(r)) + \mathrm{Vol}(B_q(\pi-r)) \leq \mathrm{vol}(M), \ \forall r \in [0,\pi]$

Using Theorem 13, we have

$$\begin{aligned}
\mathrm{vol}(M) &\geq \mathrm{Vol}(B_p(r)) + \mathrm{Vol}(B_q(\pi-r)) \\
&= \frac{\mathrm{Vol}(B_p(r))}{\mathrm{Vol}(B^1(r))}\mathrm{vol}(B^1(r)) + \frac{\mathrm{Vol}(B_q(\pi-r))}{\mathrm{vol}(B^1(\pi-r))}\mathrm{vol}(B^1(\pi-r)) \\
&\overset{\text{Thm 13}}{\geq} \frac{\mathrm{Vol}(B_p(\pi))}{\mathrm{vol}(B^1(\pi))}\mathrm{vol}(B^1(r)) + \frac{\mathrm{Vol}(B_q(\pi))}{\mathrm{Vol}(B^1(\pi))}\mathrm{vol}(B^1(\pi-r)) \\
&= \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(B^1(\pi))}\big(\underbrace{\mathrm{vol}(B^1(r)) + \mathrm{vol}(B^1(\pi-r))}_{=\mathrm{vol}(B^1(\pi))}\big) \\
&= \mathrm{Vol}(M).
\end{aligned}$$

Hence all "$\geq$" are "$=$". Particulary,

$$\frac{\mathrm{Vol}(B_p(r))}{\mathrm{vol}(B^1(r))} = \frac{\mathrm{Vol}(B_p(\pi))}{\mathrm{vol}(B^1(\pi))} \quad \forall r \in (0,\pi].$$

Let $r \to 0$, we have $1 = \dfrac{\mathrm{vol}(B_p(\pi))}{\mathrm{vol}(B^1(\pi))} = \dfrac{\mathrm{Vol}(M)}{\mathrm{vol}(S^n(1))}$.

Then Cor. 9 implies that $M$ is isometric to $S^n(1)$. $\square$

(ii) $\boxed{\text{Volume growth rate estimate}}$

<u>Theorem 15</u>. Let $(M^n, g)$ be a complete Riemannian manifold with $Ric \geq 0$.

(1) we have $Vol(B_p(r)) \leq Vol(B^0(r)) = \omega_n r^n$, where $\omega_n = Vol(B^0(1))$,
and "=" holds iff $M$ is isometric to $\mathbb{R}^n$.

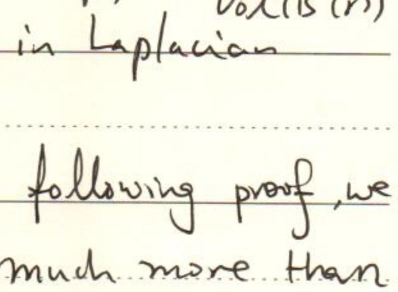
(2) If, furthermore, $M^n$ is non-compact, then there exists a positive constant $c$ depending only on $p$ and $n$ such that
$$Vol(B_p(r)) \geq c r$$
for any $r > 2$.

( Calabi, Notices AMS 1975, 丘成桐, Indiana Univ. Math J. 1976
independently. )

<u>Proof</u>. (1) follows directly from Thm 12, and $\lim_{r\to 0} \frac{Vol(B_p(r))}{Vol(B^0(r))} = 1$.
The "=" case again follows from that in Laplacian comparison Theorem.

(2). (Following <u>Gromov</u>). From the following proof, we see again that $\frac{Vol(B_p(r))}{Vol(B^n(r))} \downarrow$ tells much more than only $Vol(B_p(r)) \leq Vol(B^n(r))$ !!

Since $M$ is noncompact complete, for any $p \in M$, there exists a <u>ray</u>, i.e., a geodesic $\gamma : [0, \infty) \to M$ with
$$\gamma(0) = p, \text{ and } d(p, \gamma(t)) = t, \ \forall t \geq 0.$$
(Excercise: Prove the above claim).



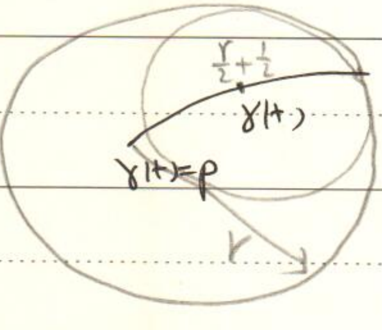

Let $t > \frac{3}{2}$, then Thm 12 tells
$$\frac{Vol(B_{\gamma(t)}(t+1))}{Vol(B_{\gamma(t)}(t-1))} \leq \frac{\omega_n (t+1)^n}{\omega_n (t-1)^n} = \frac{(t+1)^n}{(t-1)^n}$$
On the other hand,

$$\frac{Vol(B_p(1))}{Vol(B_{\gamma(t)}(t-1))} \le \frac{Vol(B_{\gamma(t)}(t+1)) - Vol(B_{\gamma(t)}(t-1))}{Vol(B_{\gamma(t)}(t-1))} \le \frac{(t+1)^n - (t-1)^n}{(t-1)^n}.$$

i.e. $$Vol(B_{\gamma(t)}(t-1)) \ge \frac{1}{t} \frac{(t-1)^n}{(t+1)^n - (t-1)^n} Vol(B_p(1))\, t \,.$$

Observe that $\exists\ C_n > 0$ s.t. $\dfrac{(t-1)^n}{t\left((t+1)^n - (t-1)^n\right)} \ge C_n$ on $\left[\frac{3}{2}, \infty\right)$.

Since $B_p(r) \supset B_{\gamma(\frac{r}{2}+\frac{1}{2})}\left(\frac{r}{2} - \frac{1}{2}\right)$ $\forall\, r > 2$.




$$\Rightarrow Vol(B_p(r)) \ge Vol\left(B_{\gamma(\frac{r}{2}+\frac{1}{2})}\left(\frac{r}{2} - \frac{1}{2}\right)\right)$$

$$\ge C_n\, Vol(B_p(1)) \cdot \left(\frac{r}{2} + \frac{1}{2}\right). \qquad \square$$

Final Remark. Recall we used Laplacian comparison to prove the volume comparison: $Ric \ge (n-1)k \Rightarrow Vol(B_p(r)) \le Vol(B^k(r))$. (Inside the cut locus, this is due to Bishop. Gromov made the crucial step to show it for any $r$, and to a "full" use of the fact $\dfrac{Vol(B_p(r))}{Vol(B^k(r))} \downarrow$.)

It is natural to ask the "other direction". Recall we do not have the "other direction" in Laplacian comparison, but we do have it in Hessian Laplacian.

Exercise (Günther. 1960) Let $(M,g)$ be a complete Riemannian manifold, ~~and~~ with ~~$Ric \ge (n-1)k$~~ sectional curvature $\le k$. Let $B_p(r)$ be a ball in $M$ which does not meet the cut locus of $p$. Then
$$vol(B_p(r)) \ge vol(B^k(r)) \qquad \square$$