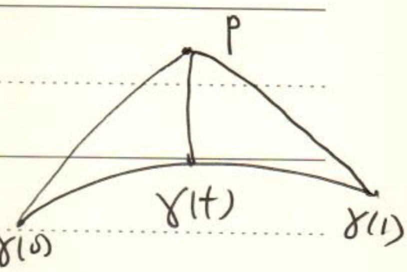
(VII). Candidates for synthetic curvature conditions

In the last part of our course, we discuss ~~some~~ properties of a Rie. mfld which does not necessarily depend on the smooth structure of the underlying spaces. Those properties may be taken to be definition of a ~~general~~ (metric, measure) space with curvature restrictions.

1. Nopositive sectional curvature and Convexity.

Theorem 1. Let (M,g) be a complete, simply-connected Rie. mfld with nonpositive sectional curvature. ~~Then~~ Let $p\in M$, ~~$\rho(x) := d(x,p), \forall x\in M$~~

$\gamma : [0,1] \longrightarrow M$ be a geodesic.



Then

$$d^2(p,\gamma(t)) \leq (1-t)\, d^2(p,\gamma(0)) + t\, d^2(p,\gamma(1)) - t(1-t)\, d^2(\gamma(0),\gamma(1)). \quad (***)$$

~~Proof~~ Remark: Actually, on a complete Rie. mfld with $\sec \leq 0$, $(***)$ holds whenever $\gamma(t) \subset E_p$. For simplicity, we suppose M is simply-connected, then ~~$\gamma(t)$~~ $E_p = \emptyset$. and $\exp_p : T_pM \to M$ is a diffeomorphism.

Proof. Let $k_0 : [0,1] \to \mathbb{R}$ be given by

$$k_0(t) = (1-t)\, d^2(p,\gamma(0)) + t\, d^2(p,\gamma(1)) - t(1-t)\, d^2(\gamma(0),\gamma(1)).$$

We have $k_0(0) = d^2(p,\gamma(0))$, $k_0(1) = d^2(p,\gamma(1))$

$$k_0''(t) = -2\, d^2(\gamma(0),\gamma(1)) = 2\,|\dot\gamma(t)|^2.$$

Let $\rho(x) := d^2(x,p)$, then $\rho\circ\gamma(t)$ satisfies

$$\rho\circ\gamma(0) = d^2(p,\gamma(0)) = k_0(0)$$

$$\rho\circ\gamma(1) = d^2(p,\gamma(1)) = k_0(1)$$

Corollary 7. (p.267)

$$\rho\circ\gamma''(t) = \operatorname{Hess}\rho\,(\dot\gamma(t),\dot\gamma(t)) \geq 2\,|\dot\gamma(t)|^2 = k_0''(t).$$

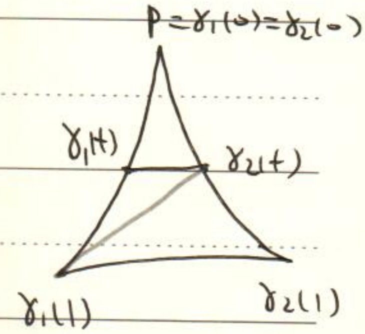
Therefore the function $h: [0,1] \to \mathbb{R}$ given by

$$h(t) := (\rho\circ\gamma - k_0)(t).$$

satisfies that $h(0) = h(1) = 0$, $h''(t) \geq 0$, $\forall t \in [0,1]$.

Therefore $h(t) \leq 0$. (Convex fcts attains maximum on the boundary)

That is $\rho\circ\gamma(t) \leq k_0(t)$, $\forall t \in [0,1]$. □



Corollary 1. Let (M,g) be a cpl. simply-connected Rie. mfld with sec ≤ 0. Let

$\gamma_1, \gamma_2: [0,1] \to M$

be geodesics with $\gamma_1(0) = p = \gamma_2(0)$.

Then for $0 \leq t \leq 1$,

$$d(\gamma_1(t), \gamma_2(t)) \leq t\, d(\gamma_1(1), \gamma_2(1)).$$

Proof: Applying Theorem 1 twice:

① $d^2(\gamma_1(1), \gamma_2(t)) \leq t d^2(\gamma_1(1), \gamma_2(1)) + (1-t)\, d^2(\gamma_1(1), p)$
$$- t(1-t)\, d^2(p, \gamma_2(1))$$

② $d^2(\gamma_2(t), \gamma_1(t)) \leq t d^2(\gamma_1(1), \gamma_2(t)) + (1-t)\, d^2(\gamma_2(t), p)$
$$- t(1-t)\, d^2(p, \gamma_1(1)).$$

Inserting ① into ②, and observing $d^2(\gamma_2(t), p) = t^2 d^2(\gamma_1(t), p)$, we complete the proof. □

Remark: The property (***) for all p and all geodesic γ is actually equivalent to the nonpositive sectional curvature of M. Namely, if the sectional curvature $\geq k > 0$ in a neighborhood of p, then locally $(\rho\circ\gamma)''(t) = \mathrm{Hess}\rho(\dot\gamma(t), \dot\gamma(t)) \leq \mathrm{Hess}\bar\rho(\dot{\bar\gamma}(t), \dot{\bar\gamma}(t))$ in $S(\frac{1}{\sqrt{k}})$.

Then one can show ">" in (***).

In fact, this is taken to be the definition of a length space with nonpositive sectional curvature in the sense of Alexandrov.

Cor 1. is also equivalent to nonpositive sectional curvature, and is taken ~~to~~ as a general curvature bound ~~b~~ notion by Busemann.
see [Chap 2. Jost, Nonpositive Curvature: Geometric and Analytic Aspects, Birkhäuser].

Excercise. Let (M,g) be a cpl. simply-connected Rie. mfld with $\sec \le 0$. Let $\gamma_1, \gamma_2 : [0,1] \to M$.
Then $d(\gamma_1(t), \gamma_2(t))$ is a convex function of $t \in [0,1]$.

Theorem 2 (Reshetnyak's quadrilateral comparison theorem).
Let (M,g) be a cpl. simply-connected Rie. mfld with $\sec \le 0$.
Let $\gamma_1, \gamma_2 : [0,1] \to M$. ~~with~~
Then

$$
\begin{aligned}
& d^2(\gamma_1(0), \gamma_2(1)) + d^2(\gamma_2(0), \gamma_1(1)) \\
\le\ & d^2(\gamma_1(0), \gamma_2(0)) + d^2(\gamma_1(1), \gamma_2(1))^2 \\
& + d^2(\gamma_1(0), \gamma_1(1)) + d^2(\gamma_2(0), \gamma_2(1)) \\
& - (d(\gamma_1(0), \gamma_2(0)) - d(\gamma_1(1), \gamma_2(1)))^2
\end{aligned}
$$

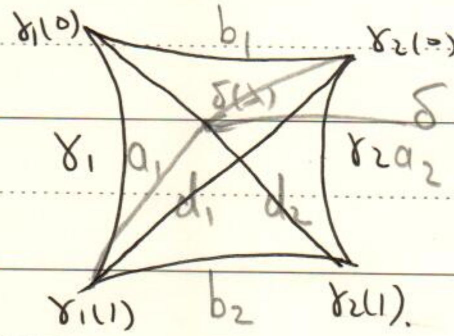


Fig. 1

Proof. For simplicity, denote the ~~length~~ distance ~~b~~ as $a_1, a_2, b_1, b_2, d_1, d_2$ as in the Fig 1.
Let δ be ~~a~~ the geodesic from $\gamma_1(0)$ to $\gamma_2(1)$ whose length is d_2.
Consider

$$d_\lambda^{2,0} := d(\gamma_2(0), \delta(\lambda))$$

$$d_\lambda^{1,1} = d(\gamma_1(1), \delta(\lambda))$$

Thm 1 $\Rightarrow$

$$(d_\lambda^{2,0})^2 \le (1-\lambda) b_1^2 + \lambda a_2^2 - \lambda(1-\lambda) d_2^2$$

$$(d_\lambda^{1,1})^2 \le (1-\lambda) a_1^2 + \lambda b_2^2 - \lambda(1-\lambda) d_2^2 .$$

Therefore, for $\varepsilon > 0$,

$$d_1^2 \le \left(d_\lambda^{2,0} + d_\lambda^{1,1}\right)^2 \le (1+\varepsilon)\left(d_\lambda^{2,0}\right)^2 + \left(1+\frac{1}{\varepsilon}\right)\left(d_\lambda^{1,1}\right)^2$$

$$\le (1+\varepsilon)(1-\lambda) b_1^2 + (1+\varepsilon)\lambda a_2^2$$

$$+ \left(1+\frac{1}{\varepsilon}\right)(1-\lambda) a_1^2 + \left(1+\frac{1}{\varepsilon}\right)\lambda b_2^2$$

$$- \left(2+\varepsilon+\frac{1}{\varepsilon}\right)\lambda(1-\lambda) d_2^2 .$$

Set $\varepsilon = \frac{1-\lambda}{\lambda}$ so that the coeff. of d_2^2 becomes

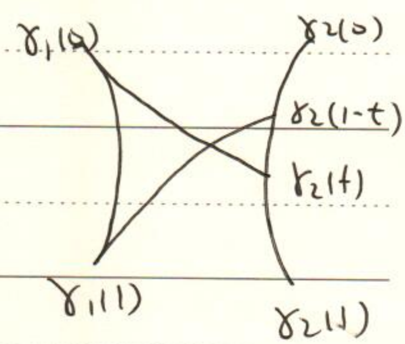
$$\left(2 + \frac{1-\lambda}{\lambda} + \frac{\lambda}{1-\lambda}\right)\lambda(1-\lambda) = 2\lambda(1-\lambda) + (1-\lambda)^2 + \lambda^2$$
$$= 1$$

This yields $d_1^2 + d_2^2 \le \frac{1}{\lambda}(1-\lambda) b_1^2 + a_2^2 + a_1^2 + \frac{\lambda}{1-\lambda} b_2^2$.

Set $\lambda = \frac{b_1}{b_1+b_2} \Rightarrow 1-\lambda = \frac{b_2}{b_1+b_2} \Rightarrow \frac{1-\lambda}{\lambda} = \frac{b_2}{b_1}$

$$\Rightarrow d_1^2 + d_2^2 \le a_1^2 + a_2^2 + 2 b_1 b_2$$
$$= a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1 - b_2)^2 \qquad \square$$

<u>Corollary 2</u>. Let (M,g) be as in Thm 2, and $\gamma_1, \gamma_2 : [0,1] \to M$ be geodesics. Then $\forall\, 0 \le t \le 1$, $0 \le s \le 1$



$$d^2(\gamma_1(0), \gamma_2(t)) + d^2(\gamma_1(1), \gamma_2(1-t))$$
$$\le d^2(\gamma_1(0), \gamma_2(0)) + d^2(\gamma_1(1), \gamma_2(1))$$
$$+ 2t^2 d^2(\gamma_2(0), \gamma_2(1))$$
$$+ t\left(d^2(\gamma_1(0), \gamma_1(1)) - d^2(\gamma_2(0), \gamma_2(1))\right)$$
$$- ts\left(d(\gamma_1(0), \gamma_1(1)) - d(\gamma_2(0), \gamma_2(1))\right)^2$$
$$- t(1-s)\left(d(\gamma_1(0), \gamma_2(0)) - d(\gamma_1(1), \gamma_2(1))\right)^2 . \qquad (\clubsuit)$$

<u>Proof</u>. Notice that Thm 2 is the case $t=1$, $s=0$.

By symmetry, we also have $(\clubsuit)$ holds for $t=1$, $s=1$

i.e. $d_1^2 + d_2^2 \le a_1^2 + a_2^2 + b_1^2 + b_2^2 - (a_1 - a_2)^2$.

Taking convex combinations yields the ineq. for $t=1$, $0\le s\le 1$:

$$d_1^2+d_2^2 \le a_1^2+a_2^2+b_1^2+b_2^2 - s(a_1-a_2)^2-(1-s)(b_1-b_2)^2.$$

Therefore, for $0\le t\le 1$, Thm 1 implies

$$
\begin{aligned}
& d^2(\gamma_1(0),\gamma_2(t)) + d^2(\gamma_1(t),\gamma_2(1-t)) \\
\le\ & (1-t)b_1^2 + t d_2^2 - t(1-t)a_2^2 \\
& + t d_1^2 + (1-t)b_2^2 - t(1-t)a_2^2 \\
\le\ & t\left(a_1^2+a_2^2+b_1^2+b_2^2 - s(a_1-a_2)^2-(1-s)(b_1-b_2)^2\right) \\
& + (1-t)(b_1^2+b_2^2) - 2t(1-t)a_2^2 \\
=\ & b_1^2+b_2^2+2t^2a_2^2+\underbrace{ta_1^2-ta_2^2}_{t(a_1^2-a_2^2)} - ts(a_1-a_2)^2 - t(1-s)(b_1-b_2)^2.
\end{aligned}
$$

□

Excercise: Let (M,g) be as above, $\gamma_1,\gamma_2:[0,1]\to M$ be geodesics. Then we have $\forall\ 0\le t\le 1$, $0\le s\le 1$.

$$
\begin{aligned}
d^2(\gamma_1(t),\gamma_2(t)) \le\ & (1-t)d^2(\gamma_1(0),\gamma_2(0)) + t d^2(\gamma_1(1),\gamma_2(1)) \\
& - t(1-t)\{ s\,(d(\gamma_1(0),\gamma_1(1)) - d(\gamma_2(0),\gamma_2(1)))^2 \\
& + (1-s)(d(\gamma_1(0),\gamma_2(0)) - d(\gamma_1(1),\gamma_2(1)))^2 \}
\end{aligned}
$$

Hint: Using ~~th~~ Cor. 2.

Remark: The above Excercise tells in particular that

$$d^2 : M\times M \to \mathbb{R}$$

where M is a cpl. simply-connected Rie. mfld with $\sec\le 0$, is a convex function. This is because a geodesic γ on $M\times M$ is given as (γ_1,γ_2) where γ_1,γ_2 are geodesics in M. Excercise tells that $d^2\circ\gamma = d^2(\gamma_1(t),\gamma_2(t))$ is a convex function ([JJ, Corollary 4.8.2]).