2. Bochner technique and Bakry-Émery Γ-Calculus

$\boxed{\text{A computation trick}}$: When we verify tensor equalities or tensor inequalities, we can pick a proper local coordinate or a proper local frame at a point, and check the equality or inequality at the point.

- good local coordinate : normal coordinate system around $p \in M$: $x : U \ni p \to \mathbb{R}^n$

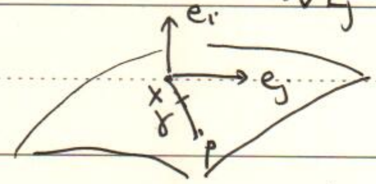
$$\begin{cases} x^i(p) = 0, \forall i \\ g_{ij}(p) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(p) = \delta_{ij}, \forall i,j \\ \Gamma^i_{jk}(p) = 0 \end{cases}$$

- (local) coordinate $\underset{\uparrow}{\underline{\underline{\text{frame}}}}$: $\left\{\frac{\partial}{\partial x^i}\right\}$

  a (local) basis of the space of vector fields.

  (local) orthonormal frame: $\{e_i\}$ with $\langle e_i, e_j\rangle = \delta_{ij}$.

  local $\underline{\text{normal}}$ frame at x: $\{E_i\}$ with $\nabla_{E_i} E_j(x) = 0$, $\forall i,j$.
  (法标架场)
  $\Rightarrow \nabla_v E_j(x) = 0$, $\forall j$, $\forall v \in T_xM$.



Existence: Pick an orthonormal basis $\{e_1, \cdots, e_n\}$ for the vector space T_xM.

Choose any $p \in M$ near to x, let γ be the geodesic from x to p. Let $E_i(p)$ be the vector in T_pM which is obtained by transport e_i parallelly along γ. The the local frame obtained in this way is what we want.

Let us discuss an example.

Lemma 1: Choosing any local frame $\{V_i\}_{i=1}^n$ on M and its dual coframe $\{\omega^i\}_{i=1}^n$, then we have

$$d = \sum_i \omega^i \wedge \nabla_{V_i} \qquad (\ast 1)$$

Recall $d: A^p(M) \to A^{p+1}(M)$ is the exterior derivative where $A^p(M)$ stands for the ~~set~~ vectorspace of smooth p-forms on M.

Proof: Let us denote by $\tilde{d} := \sum_i \omega^i \wedge \nabla_{v_i}$. Notice this does not depend on the choice of different frames. Indeed, for the frame $X_k = c_k^i V_i$ and its dual $\eta^k = d_i^k \omega^i$ where $\sum_k c_k^i d_j^k = \delta_j^i$.

Hence
$$\sum_k \eta^k \wedge \nabla_{X_k} = \sum_{k,j,i} d_j^k \omega^j \wedge \nabla_{c_k^i V_i}$$
$$= \sum_{k,j,i} d_j^k c_k^i \, \omega^j \wedge \nabla_{V_i} = \sum_{j,i} \delta_j^i \omega^j \wedge \nabla_{V_i} = \sum_i \omega^i \wedge \nabla_{V_i}.$$

As both sides of (✠1) are independent of the choice of frames, we can prove it pointwise, and choose the normal coordinate $\{x^i\}$ around a fixed point $p \in M$, and consider the local coordinate frame $\{\frac{\partial}{\partial x^i}\}$ and its dual $\{dx^i\}$.

First, we observe
$$\left(\nabla_{\frac{\partial}{\partial x^i}} dx^j\right)\left(\frac{\partial}{\partial x^k}\right) = \nabla_{\frac{\partial}{\partial x^i}}\left(dx^j\left(\frac{\partial}{\partial x^k}\right)\right) - dx^j\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\right)$$
$$= -dx^j\left(\Gamma_{ik}^l \frac{\partial}{\partial x^l}\right) = -\Gamma_{ik}^l \delta_l^j = -\Gamma_{ik}^j.$$

This means that $\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma_{ik}^j dx^k$.

Hence, at p
$$\left(\nabla_{\frac{\partial}{\partial x^i}} dx^j\right)(p) = 0. \qquad (✠2)$$

By the linear property of d, and RHS of (✠1)

We only need to verify (✠1) when applying to any q-form
$$\eta = f\, dx^1 \wedge \cdots \wedge dx^q$$

Then
$$\left(\sum_i dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}\right)\eta = \sum_i dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}\left(f\, dx^1 \wedge \cdots \wedge dx^q\right)$$
$$\overset{(✠2)}{=} \sum_i dx^i \wedge \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^q = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge dx^q$$
$$= d\eta.$$
□

$\boxed{\text{The Hodge Laplacian}}$ On a Rie. mfld (M,g), g induces an inner product on T_xM for each $x\in M$: Let us denote

$$g(X,Y) = \langle X,Y\rangle, \ \forall X,Y\in T_xM.$$

Choose a local orthonormal frame $\{E_i\}_{i=1}^n$ on M, i.e. $\langle E_i,E_j\rangle=\delta_{ij}$. Let $\{\omega^i\}_{i=1}^n$ be the dual coframe of $\{E_i\}_{i=1}^n$, i.e. $\omega^i(X_j)=\delta^i_j$. We stipulate that these 1-forms are orthonormal pairwise, i.e. we define $\langle\omega^i,\omega^j\rangle:=\delta^{ij}$.

Extend it ~~by~~ linearly: $\forall\ \varphi,\psi\in A^1(M)$, suppose

$$\varphi=\sum_i\varphi_i\omega^i,\quad \psi=\sum_j\psi_j\omega^j.$$

Then

$$\langle\varphi,\psi\rangle:=\sum_i\varphi_i\psi_i.$$

We can check that the above definition is independent of the choice of frames. In this way, each cotangent space T_x^*M becomes an $\boxed{\text{inner product space}}$.

We can continue to assign a natural inner product on the space $\Lambda^pT_x^*M$: We stipulate that the p-forms $\{\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}: i_1<\cdots<i_p\}$ are orthonormal. ~~at each point at each point~~.

$\forall\ \varphi,\psi\in A^p(M)$, suppose $\varphi=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$

$$\psi=\sum_{i_1<\cdots<i_p}\psi_{i_1\cdots i_p}\omega^{i_1}\wedge\cdots\wedge\omega^{i_p},$$

then define

$$\langle\varphi,\psi\rangle(x):=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}\psi_{i_1\cdots i_p}(x).$$

We can check that this definition is independent of the choice of frame.

Now Let M be a n-dimensional ~~orientable~~ orientable manifold and we suppose $\{\omega^i\}$ is a locally orthonormal coframe such that $\omega^1\wedge\cdots\wedge\omega^n$

is the volume form Ω on M. Recall Ω determines an orientation on M.
Then we define the **Hodge star operator** $* : A^p(M) \to A^{n-p}(M)$ $\forall\ p = 0, \dots, n$.
by requiring

$$\boxed{\varphi \wedge * \psi = \langle \varphi, \psi \rangle \Omega, \ \forall\ \varphi, \psi \in A^p(M)}$$

For that purpose, ~~we~~ observe that we have to ~~req~~ define

$$*(\omega^1 \wedge \cdots \wedge \omega^p) := \omega^{p+1} \wedge \cdots \wedge \omega^n. \qquad (1)$$

If $1 \le i_1 < \cdots < i_p \le n$, $1 \le i_{p+1} < \cdots < i_n \le n$, and

$$\{ i_{p+1}, \dots, i_n \} = \{1, \dots, n\} \setminus \{i_1, \dots, i_p\},$$

then $\{ \omega^{i_1}, \dots, \omega^{i_p}, \omega^{i_{p+1}}, \dots, \omega^{i_{n-1}}, \varepsilon_{i_1 \cdots i_n} \omega^{i_n} \}$
is also an orthonormal coframe determining the same orientation,
where $\varepsilon_{i_1, \dots, i_n}$ is the sign of the permutation

$$(1, \dots, n) \to (i_1, \dots, i_n).$$

$$\left(\begin{array}{l} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_{n-1}} \wedge \varepsilon_{i_1 \cdots i_n} \omega^{i_n} \\ = \omega^1 \wedge \cdots \wedge \omega^n = \Omega. \end{array} \right).$$

Hence, by the definition (1), we have

$$*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = \varepsilon_{i_1 \cdots i_n} \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n}.$$

Extending $*$ as an $A^0(M)$-linear operator, we have for

$$f = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

we have

$$* f := \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} * (\omega^{i_1} \wedge \cdots \wedge \omega^{i_p})$$

$$= \sum_{\substack{i_1 < \cdots < i_p \\ i_{p+1} < \cdots < i_n}} \varepsilon_{i_1 \cdots i_n} f_{i_1 \cdots i_p} \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n}.$$

We can check that this definition ~~a~~ is independent of the choice of an orthonormal ~~c~~oframe. Moreover we have for

$$\varphi = \sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\psi = \sum_{i_1 < \cdots < i_p} \psi_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

that $\varphi \wedge * \psi = \left(\sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}\right) \wedge \left(\sum_{\substack{j_1 < \cdots < j_p \\ j_{p+1} < \cdots < j_n}} \varepsilon_{j_1 \cdots j_n} \psi_{j_1 \cdots j_p} \omega^{j_{p+1}} \wedge \cdots \wedge \omega^{j_n}\right)$

$$= \sum_{\substack{i_1 < \cdots < i_p \\ i_{p+1} < \cdots < i_n}} \varphi_{i_1 \cdots i_p} \psi_{i_1 \cdots i_p} \varepsilon_{i_1 \cdots i_n} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n}$$

$$= \left(\sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} \psi_{i_1 \cdots i_p}\right) \omega^1 \wedge \cdots \wedge \omega^n = \langle \varphi, \psi \rangle \Omega. \quad \square$$

<u>Proposition 1</u>. We have $*\Omega = 1$, $*1 = \Omega$

$**\varphi = (-1)^{p(n-p)} \varphi$ for $\varphi \in A^p(M)$.

$\langle \varphi, \psi \rangle = \langle *\varphi, *\psi \rangle$, $\forall \varphi, \psi \in A^p(M)$.

<u>Proof</u>: We only show the last ~~two~~:

$$**\varphi = *\left(\sum_{\substack{i_1 < \cdots < i_p \\ i_{p+1} < \cdots < i_n}} \varphi_{i_1 \cdots i_p} \varepsilon_{i_1 \cdots i_n} \omega^{i_{p+1}} \wedge \cdots \wedge \omega^{i_n}\right)$$

$$= \sum_{\substack{i_1 < \cdots < i_p \\ i_{p+1} < \cdots < i_n}} \varphi_{i_1 \cdots i_p} \underbrace{\varepsilon_{i_1 \cdots i_n} \varepsilon_{i_{p+1} \cdots i_n i_1 \cdots i_p}} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$= \sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} (-1)^{p(n-p)} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$$

$$\left(\operatorname{sgn}\left((1, \cdots, n) \to (i_1 \cdots, i_n)\right) \cdot \operatorname{sgn}\left((1, \cdots, n) \to (i_{p+1} \cdots i_n i_1 \cdots i_p)\right) = (-1)^{(n-p)p}\right)$$

$$= (-1)^{p(n-p)} \varphi. \quad \square$$

Then $\langle *\varphi, *\psi \rangle \Omega = *\varphi \wedge *(*\psi) = (-1)^{p(n-p)} *\varphi \wedge \psi$

$$= \psi \wedge *\varphi = \langle \psi, \varphi \rangle \Omega = \langle \varphi, \psi \rangle \Omega.$$

Hence $\langle *\varphi, *\psi \rangle = \langle \varphi, \psi \rangle$. $\square$

<u>Definition 1</u> (Hodge Laplacian) We define the operator

$$\delta : A^p(M) \to A^{p-1}(M).$$

by $\delta := (-1)^{np+n+1} *d*$. The Hodge Laplacian Δ is defined as $\Delta = \delta d + d\delta : A^p(M) \to A^p(M)$.

Let us explain why we define δ like this, especially with such a complicated sign. Let M be an n-dim'l, $\boxed{\text{closed}}$, orientable, Rie. mfld.

We can introduce an inner product in the space of the whole exterior Algebra $A^*(M) := \oplus_{p=0}^{n} A^p(M)$ as below:

$\forall\, \varphi, \psi \in A^p(M)$, set

$$(\varphi, \psi) := \int_M \langle \varphi, \psi \rangle \Omega = \int_M \varphi \wedge * \psi$$

$\forall\, \varphi \in A^p(M), \psi \in A^q(M)$, with $p \neq q$,

$$(\varphi, \psi) := 0.$$

Excercise: Check this definition gives an inner product in A^*.

Proposition 1 implies

$$\cancel{\langle *\varphi, *\psi \rangle =} \; (*\varphi, *\psi) = (\varphi, \psi).$$

That is, $*$ is an isometry transformation between A^* and itself.

The ~~δ~~ definition of δ is carefully given to ensure the following property:

Proposition 2. $\forall\, \alpha \in A^{p-1}(M)$, $\beta \in A^p(M)$, we have

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

Proof: $(d\alpha, \beta) = \int_M d\alpha \wedge *\beta \underset{\text{definition of } d.}{=} \int_M d(\alpha \wedge *\beta) - (-1)^{p-1} \alpha \wedge d*\beta$

$$\underset{\text{Stokes}}{=} -(-1)^{p-1} \int_M (-1)^{(n-p+1)(p-1)} \alpha \wedge **(\underbrace{d*\beta}_{\in A^{n-p+1}})$$

$$\underset{\text{Prop. 1}}{=} (-1)^{(n-p+2)(p-1)+1} \int_M \alpha \wedge *(*d*\beta).$$

Excercise: $(-1)^{(n-p+2)(p-1)+1} = (-1)^{np+n+1}$

Hence $(d\alpha, \beta) = \int_M \alpha \wedge *(\delta\beta) = (\alpha, \delta\beta)$. $\square$

Now the definition of δ is justified.

Notice that ~~d~~ $\delta^2 = 0$ follows from $d^2 = 0$. Hence

$$\Delta = \delta d + d\delta = (d+\delta)(d+\delta).$$

<u>Proposition 3</u> The Hodge Laplacian Δ is a self-adjoint operator.

<u>Proof</u>: $\forall\, \alpha, \beta \in A^*$, we have

$$
\begin{aligned}
(\Delta\alpha, \beta) &= ((d+\delta)(d+\delta)\alpha, \beta) \\
&= (d(d+\delta)\alpha, \beta) + (\delta(d+\delta)\alpha, \beta) \\
&= ((d+\delta)\alpha, \delta\beta) + ((d+\delta)\alpha, d\beta) \\
&= ((d+\delta)\alpha, (d+\delta)\beta) \\
&= (\alpha, (d+\delta)(d+\delta)\beta) = (\alpha, \Delta\beta). \quad \square
\end{aligned}
$$

As a direct corollary, Δ is a positive operator on $A^*(M)$.

That is, $\forall\, f \in A^*$, $(\Delta f, f) = ((d+\delta)f, (d+\delta)f)$

$$= (df, df) + (\delta f, \delta f) \geq 0.$$

Suppose λ is any eigenvalue of Δ, i.e., $\Delta f = \lambda f$ for some nontrivial f, then $\lambda(f,f) = (\Delta f, f) \geq 0$.

$$\Rightarrow \quad \lambda \geq 0.$$

Furthermore $\boxed{f \in A^* \text{ is harmonic } (\Delta f = 0) \text{ iff } df = \delta f = 0}$.

Next, we aim at establishing an expression for δ similar to Lemma 1 for d. Recall for any vector field X on M, the <u>interior product</u> (内積)

$$i(X) : A^p(M) \to A^{p-1}(M)$$

by $(i(X)\varphi)(Y_1, \dots, Y_{p-1}) = \varphi(X, Y_1, \dots, Y_{p-1})$. $\forall\, \varphi \in A^p(M)$, and any v.f. $Y_1, \dots, Y_{p-1}$.

<u>Proposition 4</u>. For any $\varphi \in A^p(M)$, $\psi \in A^q(M)$, we have

(i) $i(X)(\varphi \wedge \psi) = (i(X)\varphi) \wedge \psi + (-1)^p \varphi \wedge (i(X)\psi)$.

(ii) $i(X)(f\varphi) = f(i(X)\varphi)$.

(iii) $i(X) \circ i(X) = 0$.

<u>Proof</u>: Direct proof. We check (iii). $\forall\, \varphi \in A^p(M)$, we have

$$((i(X) \circ i(X))\varphi)(Y_1, \dots, Y_{p-2}) = \varphi(X, X, Y_1, \dots, Y_{p-2}) = 0 \quad \square$$

Lemma 2. Choosing any local orthonormal frame $\{E_i\}_{i=1}^n$ on M, we have

$$\delta = -\sum_{j=1}^{n} i(E_j)\nabla_{E_j}. \qquad (**)$$

Proof: Denote $\bar{\delta} = -\sum_{j=1}^{n} i(E_j)\nabla_{E_j}$. We can check $\bar{\delta}$ is independent of the choice of local orthonormal frames. So we only need to prove $(**)$ at a fixed point $x \in M$. We pick a local normal frame $\{E_i\}$ at x. Let $\{\omega^i\}$ be the dual coframe.

Since $\quad \nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$

we have $\quad (\nabla_{E_i}\omega^j)(E_k) = \nabla_{E_i}(\omega^j(E_k)) - \omega^j(\nabla_{E_i}E_k)$

$$= -\omega^j(\Gamma_{ik}^l E_l) = -\Gamma_{ik}^j$$

$$\Rightarrow \quad \nabla_{E_i}\omega^j = -\Gamma_{ik}^j \omega^k.$$

Therefore at $x \in M$, we have $\quad \nabla_{E_i}\omega^j(x) = 0, \ 1 \le i, j \le n.$

By linearity, we only need to verify

$$\delta\eta = \bar{\delta}\eta \quad \text{for } \eta = f\,\omega^1\wedge\cdots\wedge\omega^p.$$

We compute $\quad \bar{\delta}\eta = -\sum_{j=1}^{p} i(E_j)\nabla_{E_j}(f\,\omega^1\wedge\cdots\wedge\omega^p)$

$$= -\sum_{j=1}^{p} i(E_j)(E_j(f))\,\omega^1\wedge\cdots\wedge\omega^p$$

$$= -\sum_{j=1}^{p} E_j(f)\sum_{k=1}^{p}(-1)^{k-1}\omega^1\wedge\cdots\wedge\omega^{k-1}\wedge i(E_j)\omega^k\wedge\cdots\wedge\omega^p$$

$$= -\sum_{j=1}^{p} E_j(f)\,(-1)^{j-1}\,\omega^1\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^p$$

$$= \sum_{j=1}^{p}(-1)^j E_j(f)\,\omega^1\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^p.$$

On the other hand, note that for $j = 1, \cdots, p$.

$$*(\omega^j\wedge\omega^{p+1}\wedge\cdots\wedge\omega^n) = (-1)^{(n-p)(p-1)}\cdot(-1)^{j-1}\,\omega^1\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^p$$

$$= (-1)^{np+n+1+j}\,\omega^1\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^p.$$

Therefore.
$$
\begin{aligned}
(\delta\eta)(p) &= (-1)^{np+n+1} * d * (f\, \omega^1 \wedge \cdots \wedge \omega^p) \\
&= (-1)^{np+n+1} * d\, (f\, \omega^{p+1} \wedge \cdots \wedge \omega^n) \\
&= (-1)^{np+n+1} * \Big(\sum_{j=1}^{n} \omega^j \wedge \nabla_{E_j} (f\, \omega^{p+1} \wedge \cdots \wedge \omega^n) \Big) \\
&= (-1)^{np+n+1} * \Big(\sum_{j=1}^{n} E_j(f)\, \omega^j \wedge \omega^{p+1} \wedge \cdots \wedge \omega^n \Big) \\
&= (-1)^{np+n+1} \sum_{j=1}^{n} E_j(f)\, * (\omega^j \wedge \omega^{p+1} \wedge \cdots \wedge \omega^n) \\
&= \sum_{j=1}^{n} (-1)^j E_j(f)\, \omega^1 \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \omega^p . \\
&= (\bar{\delta}\eta)(p) .
\end{aligned}
$$

This completes the proof. □

Observation: for $f \in C^\infty(M)$, we have

$$\Delta f = \delta d f = -\sum_j i(E_j) \nabla_{E_j} \cdot \Big(\sum_i \omega^i \wedge \nabla_{E_i} f \Big).$$

$E_i(f) . \omega^i$

$$= -\sum_j i(E_j)\, \underbrace{\nabla_{E_j} (E_i(f)\, \omega^i)}_{= E_j(E_i(f))\omega^i + E_i(f) \nabla_{E_j} \omega^i}$$

$$= -\sum_{j,i} E_j(E_i(f))\, \omega^i(E_j) . = -\sum_i E_i(E_i(f))$$

$$= -\operatorname{tr} \operatorname{Hess} f .$$

So Hodge Laplacian is the negative of the Laplace-Beltrami operator we defined before.

$\boxed{\text{Weitzenböck formula}}$.

For $\omega \in A^p(M)$, which can be considered as a $(0,p)$ tensor, recall $\nabla^2\omega$ is a $(0,p+2)$-tensor. Let $\{e_i\}$ be a local orthonormal frame. We define

$$\mathrm{tr}\,(\nabla^2\omega)(\cdots) := \sum_i \nabla^2\omega(\cdots, e_i, e_i)$$

One can check this definition is independent of the choice of an orthonormal frame, and

$$\mathrm{tr}\,(\nabla^2\omega) = \sum_i \left(\nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega\right)$$

(Exercise).

<u>Theorem 3</u> (Weitzenböck formula) For any $\omega \in A^p(M)$, let $\{e_i\}$ be a local orthonormal frame and $\{\omega^i\}$ its dual, then

$$\Delta\omega = \underbrace{-\,\mathrm{tr}\,(\nabla^2\omega)}_{\text{Connection Laplacian}} - \omega^i \wedge i(e_j)R(e_i,e_j)\omega.$$

<u>Proof</u>: We can check that the RHS is independent ~~of~~ the choice of orthonormal frame. So we will prove the W-formula at a point $x \in M$, and pick a local <u>normal</u> ~~coordinate~~ frame $\{E_i\}$. we again use $\{\omega^i\}$ for its dual.

Recall the covariant derivative is commutable with the contraction

i.e. $$\nabla_{E_i}\left(C\left(E_j \otimes \nabla_{E_j}\omega\right)\right) = C\,\nabla_{E_i}\left(E_j \otimes \nabla_{E_j}\omega\right)$$
$$= C\left(E_j \otimes \nabla_{E_i}\nabla_{E_j}\omega\right)$$

Hence $$\nabla_{E_i}\left(i(E_j)\nabla_{E_j}\omega\right) = i(E_j)\nabla_{E_i}\nabla_{E_j}\omega.$$

Recall $\nabla_{E_i}E_j = 0,\ \forall i,j \Rightarrow \nabla_{E_j}\omega^i = 0,\ \forall i,j$.

Therefore, we compute

$$\Delta\omega = d\delta\omega + \delta d\omega = \sum_i \omega^i \wedge \nabla_{E_i}(\delta\omega) - \sum_j i(E_j)\nabla_{E_j}(d\omega)$$

$$= -\sum_i \omega^i \wedge \nabla_{E_i}\left(\sum_j i(E_j)\nabla_{E_j}\omega\right) - \sum_j i(E_j)\nabla_{E_j}\left(\sum_i \omega^i \wedge \nabla_{E_i}\omega\right)$$

$$= -\sum_{i,j} \omega^i \wedge \nabla_{E_i}\left(i(E_j)\nabla_{E_j}\omega\right) - \sum_{j,i} i(E_j)\nabla_{E_j}\left(\omega^i \wedge \nabla_{E_i}\omega\right).$$

$$= -\sum_{i,j} \omega^i \wedge i(E_j)\nabla_{E_i}\nabla_{E_j}\omega - \sum_{j,i} i(E_j)\,\cancel{\nabla_{E_j}}\left(\omega^i \wedge \nabla_{E_j}\nabla_{E_i}\omega\right)$$

$$= -\sum_{i,j} \omega^i \wedge i(E_j)\nabla_{E_i}\nabla_{E_j}\omega - \sum_{j,i} \delta_j^i \nabla_{E_j}\nabla_{E_i}\omega + \sum_{j,i}\omega^i \wedge i(E_j)\nabla_{E_j}\nabla_{E_i}\omega$$

$$= -\sum_i \nabla_{E_i}\nabla_{E_i}\omega - \sum_{i,j}\omega^i \wedge i(E_j)\left(\nabla_{E_i}\nabla_{E_j}\omega - \nabla_{E_j}\nabla_{E_i}\omega\right)$$

$$= -\operatorname{tr}(\nabla^2\omega) - \sum_{i,j}\omega^i \wedge i(E_j)R(E_i,E_j)\omega. \qquad \square$$

<u>Corollary 3</u>: (Bochner). For any $\omega \in A^p(M)$, we have

$$-\frac{1}{2}\Delta|\omega|^2 = -\langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + F(\omega)$$

where $F(\omega) := -\langle \omega^i \wedge i(E_j)R(e_i,e_j)\omega, \omega\rangle$

<u>Remark</u>: Here we are using Hodge Laplacian. (also for $\Delta|\omega|^2$).

<u>Proof</u>: We compute in local normal frame $\{E_i\}$, we have

$$-\langle \Delta\omega, \omega\rangle + F(\omega) = \langle -\Delta\omega - \omega^i \wedge i(E_j)R(E_i,E_j)\omega, \omega\rangle$$

$$\overset{W\text{-formula}}{=} \langle \operatorname{tr}(\nabla^2\omega), \omega\rangle$$

$$= \langle \sum_i \nabla_{E_i}\nabla_{E_i}\omega, \omega\rangle$$

$$= \sum_i \left(\nabla_{E_i}\langle \nabla_{E_i}\omega, \omega\rangle - \cancel{2}\langle \nabla_{E_i}\omega, \nabla_{E_i}\omega\rangle\right)$$

$$= \frac{1}{2}\sum_i \nabla_{E_i}\nabla_{E_i}\langle\omega,\omega\rangle - \cancel{2}|\nabla\omega|^2$$

$$= -\frac{1}{2}\Delta|\omega|^2 - |\nabla\omega|^2. \qquad \square$$

In particular when $\omega \in A^1(M)$, the curvature term becomes simpler. Let $\#\omega = \langle\omega, \omega^i\rangle E_i$, we have

$$F(\omega) = -\langle \omega^i \wedge \underbrace{i(E_j)R(E_i,E_j)\omega}_{\in A^0(M)}, \omega\rangle$$

$$= -i(E_j)R(E_i,E_j)\omega\,\langle\omega^i, \omega\rangle$$

$$= -\left(R(E_i,E_j)\omega\right)(E_j)\,\langle\omega^i, \omega\rangle$$

<u>Claim</u>: $\big(R(X,Y)(\omega)\big)(Z) = -\omega(R(X,Y)Z)$.

$$\big(R(X,Y)(\omega)\big)(Z) = \big(\nabla_X\nabla_Y\omega - \nabla_Y\nabla_X\omega - \nabla_{[X,Y]}\omega\big)(Z)$$

$$\begin{aligned}(\nabla_X\nabla_Y\omega)(Z) &= (\nabla_X(\nabla_Y\omega))(Z) = X\big((\nabla_Y\omega)(Z)\big) - (\nabla_Y\omega)(\nabla_X Z)\\ &= X\big(Y(\omega(Z)) - \omega(\nabla_Y Z)\big) - Y\big(\omega(\nabla_X Z)\big) + \omega(\nabla_Y\nabla_X Z)\\ &= X(Y(\omega(Z))) - X(\omega(\nabla_Y Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y\nabla_X Z)\end{aligned}$$

$$(\nabla_{[X,Y]}\omega)(Z) = [X,Y](\omega(Z)) - \omega(\nabla_{[X,Y]}(Z))$$

Hence
$$\begin{aligned}\big(R(X,Y)(\omega)\big)(Z) &= \omega(\nabla_Y\nabla_X Z) - \omega(\nabla_X\nabla_Y Z) + \omega(\nabla_{[X,Y]}(Z))\\ &= -\omega(R(X,Y)Z)\end{aligned}$$

□

Now we continue our computation:

$$\begin{aligned}F(\omega) &= -\big(R(E_i,E_j)\omega\big)(E_j)\langle\omega^i,\omega\rangle\\ &= \omega\big(R(E_i,E_j)E_j\big)\langle\omega^i,\omega\rangle\\ &= \langle\#\omega, R(E_i,E_j)E_j\rangle\langle\omega^i,\omega\rangle\\ &= \langle\#\omega, R(\underbrace{\langle\omega^i,\omega\rangle E_i}_{=\#\omega}, E_j)E_j\rangle\\ &= \langle\#\omega, R(\#\omega,E_j)E_j\rangle = \langle R(\#\omega,E_j)E_j, \#\omega\rangle\\ &= \sum_j R(\#\omega,E_j,\#\omega,E_j) = \sum_j R(E_j,\#\omega,E_j,\#\omega)\\ &= \operatorname{tr}R(\cdot,\#\omega,\cdot,\#\omega) = Ric(\#\omega,\#\omega).\end{aligned}$$

<u>Corollary 4</u>: For any $\omega\in A^1(M)$, we have
$$-\frac{1}{2}\Delta|\omega|^2 = -\langle\Delta\omega,\omega\rangle + |\nabla\omega|^2 + Ric(\#\omega,\#\omega).$$

<u>Theorem 4</u>. (Bochner) Let (M,g) be a closed oriented Rie. mfld.

(1) If $Ric\geq 0$, then any harmonic 1-form ω is parallel, i.e. $\nabla\omega=0$.

(2) If $Ric\geq 0$ on M and $Ric>0$ at one point, then there is no non-trivial harmonic 1-form.

Proof: Recall $\int_M -\Delta|\omega|^2 \, dvol_M = 0$. Hence we have

$$0 = -\int_M \underbrace{\langle \Delta\omega, \omega\rangle}_{\substack{\| \\ 0 \text{ for harmonic } \omega.}} + |\nabla\omega|^2 + Ric(\#\omega, \#\omega) \, dvol_M$$

$$= \int_M |\nabla\omega|^2 + Ric(\#\omega, \#\omega) \, dvol_M \underset{\substack{\uparrow \\ Ric \geq 0.}}{\geq} 0.$$

$\Rightarrow \nabla\omega = 0$, i.e. ω is parallel.

If $Ric > 0$ at some point, we must have $\#\omega = 0$, i.e. $\omega = 0$. □

Corollary 5. For any $f \in C^\infty(M)$, we have

$$\frac{1}{2}\Delta_{LB} |\mathrm{grad} f|^2 = |\mathrm{Hess} f|^2 + \langle \mathrm{grad}(\Delta_{LB} f), \mathrm{grad} f\rangle + Ric(\mathrm{grad} f, \mathrm{grad} f).$$

Proof: We see $df \in A^1(M)$, and

$$|df|^2 = |\mathrm{grad} f|^2, \quad \#(df) = \mathrm{grad} f.$$

$$-\langle \Delta df, df\rangle = -\langle (d\delta + \delta d) df, df\rangle = -\langle d\delta df, df\rangle$$

$$= -\langle d(\delta df), df\rangle = -\langle d(\Delta f), df\rangle$$

$$= -\langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle = \langle \mathrm{grad}(\Delta_{LB} f), \mathrm{grad} f\rangle.$$

$$|\nabla df|^2 = \sum_i \langle \nabla_{E_i} df, \nabla_{E_i} df\rangle$$

$$= \sum_i \langle \nabla_{E_i} \mathrm{grad} f, \nabla_{E_i} \mathrm{grad} f\rangle$$

$$= \sum_i \Big\langle \sum_j \langle \nabla_{E_i} \mathrm{grad} f, E_j\rangle E_j, \sum_k \langle \nabla_{E_i} \mathrm{grad} f, E_k\rangle E_k \Big\rangle$$

$$= \sum_{i,j} \langle \nabla_{E_i} \mathrm{grad} f, E_j\rangle^2 = \sum_{i,j} \mathrm{Hess} f(E_i, E_j)^2$$

$$= |\mathrm{Hess} f|^2. \quad \text{(Hilbert-Schmidt norm)}. \quad \square$$

Let (M, g) be a closed Rie. mfld.
We say $\lambda \in \mathbb{R}$ is an eigenvalue of Δ_{LB} if $\exists$ a smooth function $u \neq 0$ such that $\Delta_{LB} u + \lambda u = 0$.

It is known that the eigenvalues can be listed as

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \nearrow \infty.$$

<u>Theorem 5</u> (Lichnerowicz). Let (M^n, g) be a closed Rie. mfld with $Ric \geq k > 0$. Then we have

$$\lambda_1 \geq \frac{n}{n-1} C.$$

<u>Proof</u>: Integrate the Bochner formula in Cor. 5, we have

$$0 = \int_M |\text{Hess} f|^2 + \langle \text{grad}(\Delta_{LB} f), \text{grad} f \rangle + Ric(\text{grad} f)$$

Let f be the eigenfct $\Delta_{LB} f + \lambda_1 f = 0$

$$= \int_M |\text{Hess} f|^2 - \lambda_1 \int_M \langle \text{grad} f, \text{grad} f \rangle + \int_M Ric(\text{grad} f, \text{grad} f)$$

$$\geq \int_M |\text{Hess} f|^2 - \lambda_1 \int_M |\text{grad} f|^2 + k \int_M |\text{grad} f|^2.$$

$\Rightarrow \lambda_1 \geq k.$

We can make more use of $|\text{Hess} f|^2$ term.

$$|\text{Hess} f|^2 = \sum_{i,j} \text{Hess} f(E_i, E_j)^2 \geq \sum_i \text{Hess} f(E_i, E_i)^2$$

$$\geq \frac{1}{n} \left(\sum_i \text{Hess} f(E_i, E_i) \right)^2 = \frac{1}{n} (\Delta_{LB} f)^2 = \frac{1}{n} \lambda_1^2 f^2$$

$$\Rightarrow \int_M |\text{Hess} f|^2 \geq \frac{1}{n} \lambda_1 \int_M \lambda_1 f^2 = \frac{\lambda_1}{n} \int_M \langle f, -\Delta_{LB} f \rangle$$

$$= \frac{\lambda_1}{n} \int_M \langle \text{grad} f, \text{grad} f \rangle.$$

$$\Rightarrow 0 \geq \left(\frac{\lambda_1}{n} - \lambda_1 + k \right) \int_M |\text{grad} f|^2.$$

$$\Rightarrow \lambda_1 \geq \frac{nk}{n-1}. \qquad \square$$

Like for the Bonnet – Myers Theorem, we have the following Rigidity result due to Obata.

Theorem 6 (Obata) Let (M^n, g) be a closed Rie. mfld with $Ric \geq (n-1)k$, $k>0$. Then $\lambda_1 = nk$ iff (M^n, g) is isometric to the sphere $\mathbb{S}^n\left(\frac{1}{\sqrt{k}}\right)$.

Proof: W.l.o.g., we can suppose $k=1$. If $\lambda_1 = n$, then the proof of Thm 5 implies

$$Ric(\text{grad} f, \text{grad} f) = (n-1)|\text{grad} f|^2$$

Since $\Delta_{LB}(u^2) = 2|\text{grad}\, u|^2 + 2u\Delta_{LB} u$.

$$\Rightarrow \frac{1}{2}\Delta_{LB}(|\overset{\text{grad} f}{\nabla f}|^2 + f^2) = \frac{1}{2}\Delta_{LB}|\nabla f|^2 + |\text{grad} f|^2 + f\Delta_{LB} f$$

$$\geq \frac{\overset{=n}{\lambda_1}}{n}\langle f, -\Delta_{LB} f\rangle - n|\text{grad} f|^2 + (n-1)|\text{grad} f|^2 + |\text{grad} f|^2 + f\Delta_{LB} f$$

$$= 0.$$

Recall $\int_M \frac{1}{2}\Delta_{LB}(|\nabla f|^2 + f^2) = 0$ $\Big\} \Rightarrow \Delta_{LB}(|\nabla f|^2 + f^2) = 0$

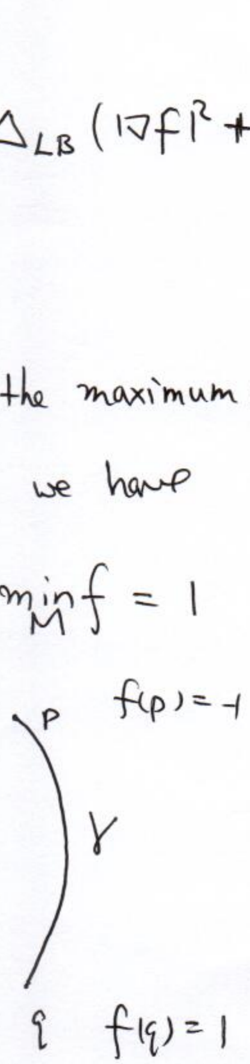
That is, $|\text{grad} f|^2 + f^2 \equiv \text{const}$.

Normalize f so that $\max_M f^2 = 1$. Since at the maximum/minimum points of f, we have $\text{grad} f = 0$. Therefore we have

$$|\text{grad} f|^2 + f^2 = 1 \quad \text{and} \quad \max_M f = -\min_M f = 1$$

Let $p, q \in M$ be points s.t.

$$f(p) = -1, \quad f(q) = 1.$$



Let γ be a normal minimizing geodesic from p to q.

Note that $$\frac{\left|\frac{d}{dt} f\circ\gamma(t)\right|}{\sqrt{1 - f\circ\gamma(t)^2}} \leq \frac{|\text{grad} f(\gamma(t))|^2}{\sqrt{1 - f(\gamma(t))^2}} = 1.$$

$0 \leq t < d(p,q)$

Integrating over t,

$$\left|\int_0^{d(p,q)} \frac{\frac{d}{dt} f\circ\gamma(t)}{\sqrt{1-f\circ\gamma(t)^2}}\right| \leq \int_0^{d(p,q)} \frac{|\frac{d}{dt} f\circ\gamma(t)|}{\sqrt{1-f\circ\gamma(t)^2}} \leq d(p,q)$$

$$\shortparallel$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \pi.$$

On the other hand, $Ric \geq (n-1) \Rightarrow d(p,q) \leq \pi$.

$$\Rightarrow \text{diam} = \pi \underset{\text{Cheng}}{\overset{}{\Rightarrow}} M \text{ isometric to } \mathbb{S}^n(1). \qquad \square$$

$\boxed{\text{Bakry - Émery } \Gamma \text{ - calculus}}$. A systematic way of understanding The Bochner formula.

For any $f, g \in C^\infty(M)$, define

$$\Gamma(f,g) := \frac{1}{2}\left(\Delta_{LB}(fg) - f\Delta_{LB} g - g\Delta_{LB} f\right)$$

Observe $\Gamma(f,f) = |\text{grad} f|^2$

Iteratively, define

$$\Gamma_2(f,g) := \frac{1}{2}\left(\Delta_{LB}(\Gamma(f,g)) - \Gamma(f, \Delta_{LB} g) - \Gamma(\Delta_{LB} f, g)\right)$$

Observe that $\Gamma_2(f,f) = \frac{1}{2}\Delta_{LB}|\text{grad} f|^2 - \langle \text{grad} f, \text{grad}(\Delta_{LB} f)\rangle$.

So the Bochner formula in Corollary 5 implies.

$$\Gamma_2(f,f) = |\text{Hess} f|^2 + Ric(\text{grad} f, \text{grad} f)$$

$$\geq \frac{1}{n}(\Delta_{LB} f)^2 + Ric(\text{grad} f, \text{grad} f).$$

Moreover $\boxed{Ric \geq K \text{ implies } \quad \Gamma_2(f,f) \geq \frac{1}{n}(\Delta_{LB} f)^2 + K\,\Gamma(f,f) \quad \forall f \in C^\infty(M).}$

$(*)$

Property ⊛ enable us to define "Ricci curvature lower bound" for a general operator, which can be operators on more general spaces. Here we discuss such possibilites on discrete metric spaces: a combinatorial graph $G = (V, E)$.

- V the set of vertices (points)
- E the set of edges

↓

metric : combinatorial distance

(length of shortest path)

For example, a discrete set $\{p_1, \cdots, p_n\}$ with the metric

$$d(p_i, p_j) = \delta_{ij}$$

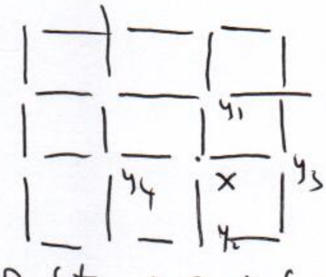
can be represented by a complete graph K_n.

Define the <u>degree</u> of a vertex p to be

$$\deg(p) := \sum_{\substack{q \in V \\ d(p,q)=1}} 1 .$$

We can consider the graph Laplacian Δ defined via

$$\Delta f(x) = \sum_{\substack{y \in V \\ d(y,x)=1}} (f(y) - f(x)).$$



for $f: V \to \mathbb{R}$.

We say λ is an eigenvalue of Δ if $\exists f \neq 0$ s.t. $\Delta f + \lambda f = 0$.

We can list

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{|V|-1} .$$

<u>Definition</u>. A Grap $G = (V, E)$ is said to satisfy the curvature dimension inequality $CD(K, n)$ for some $K \in \mathbb{R}$, $n \in (0, \infty]$ ~~at a ver~~ if for all $f: V \to \mathbb{R}$, it holds

$$\Gamma_2(f,f)(x) \geq \frac{1}{n}(\Delta f)^2(x) + K\Gamma(f,f)(x), \quad \forall x \in V.$$

Since we do not have a proper understanding about the "dimension" of a graph, quite oftenly we assume $CD(K,\infty)$ conditions.

arXiv: 1608.07778

<u>Theorem</u> (L.-Münch-Peyerimhoff, '16, '17) arXiv: 1705.08119.

Let $G=(V,E)$ be a connected graph satisfying $CD(K,\infty)$, and $deg_{max} < \infty$. Then

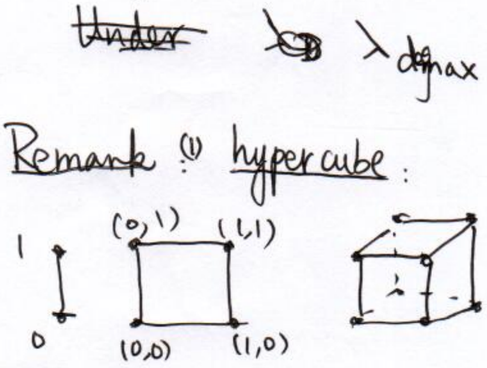
$$diam_d(G) \leq \frac{2\, deg_{max}}{K}.$$

Moreover, "=" holds if and only if G is a deg_{max}-dimensional hypercube. Under the same assumption,

(By standard argument, we have $\lambda_1 \geq K$)

~~Under~~ $\lambda_{deg_{max}} = K$ iff G is a deg_{max}-dim'l hypercube.

<u>Remark</u> (1) <u>hypercube</u>:



$\cdots \; (I)^n.$

(2) $\lambda_1 = K$ is not strong enough to conclude the Rigidity Thm.

Counterexample:

$CD(2,\infty)$
and $\lambda_1 = 2$.

<u>Open Question</u>: Let $G=(V,E)$ be a connected graph satisfying $CD(0,\infty)$. What is the volume growth rate? polynomial? This is ~~important~~ equivalent to ask for the (non-) existence of a family of expanders in the class of Graphs satisfying $CD(0,\infty)$.