staff.ustc.edu.cn/~spliu/Teaching.html.

(II) 测地线 geodesics

References: (伍鸿熙 第3章) & (Riemannian geometry and geometric analysis, 5th ed. J. Jost Section 1.4 and 1.6)

$\boxed{\text{§1. 测地线方程和Christoffel符号}}$

设 $\gamma:[a,b]\to(M,g)$ 是一条光滑曲线 (总假设参数化正则，即 $\dot\gamma(t)\neq 0,\ \forall t\in[a,b]$). 其长度定义为

$$L(\gamma):=\text{Length}(\gamma):=\int_a^b\sqrt{\langle\dot\gamma(t),\dot\gamma(t)\rangle_{\gamma(t)}}\,dt \qquad (1)$$

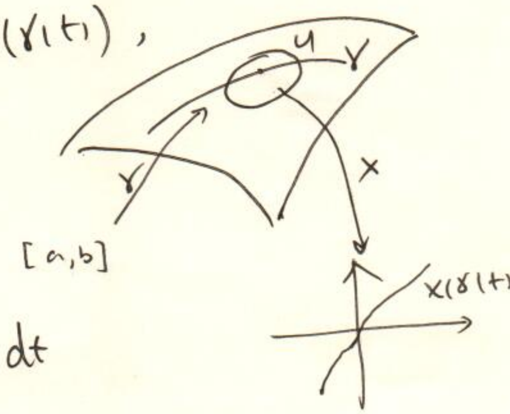
同时还可以定义 γ 的能量 (energy of γ)

$$E(\gamma):=\text{Energy}(\gamma):=\frac12\int_a^b\langle\dot\gamma(t),\dot\gamma(t)\rangle_{\gamma(t)}\,dt \qquad (2)$$

(在物理中，γ 被看作是一个质点的运动轨道，$E(\gamma)$ 又称为 "action of γ").

在局部坐标邻域 $(U,x^1,\cdots,x^n)$ $(x^1(\gamma(t)),\cdots,x^n(\gamma(t)))$,

$$[a,b]\xrightarrow{\gamma}M$$



从而 (1) 和 (2) 可写作

$$L(\gamma)=\int_a^b\sqrt{g_{ij}(x(\gamma(t)))\dot x^i(t)\dot x^j(t)}\,dt$$

$$E(\gamma)=\frac12\int_a^b g_{ij}(x(\gamma(t)))\dot x^i(t)\dot x^j(t)\,dt$$

为更好理解 $L(\gamma)$ 和 $E(\gamma)$ 的上述表示，我们看下面的例子。

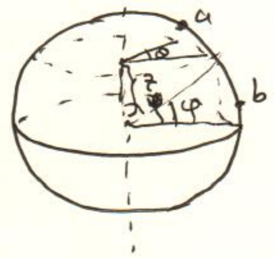
例子：考虑 $S^2\subset\mathbb{R}^3$，及坐标邻域

$$\{(\sqrt{1-z^2}\cos\theta,\ \sqrt{1-z^2}\sin\theta,\ z)\mid 0<\theta<2\pi,\ -1<z<1\}$$

坐标为 (θ,z).

第一次课中的习题：S^2上的诱导度量为

$$g=\frac{1}{1-z^2}\,dz\otimes dz+(1-z^2)\,d\theta\otimes d\theta$$



我们也可以换球坐标 (φ,θ). $0<\theta<2\pi$, $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$

那么 $z=\sin\varphi$，则此坐标下度量为

$$g=d\varphi\otimes d\varphi+\cos^2\varphi\,d\theta\otimes d\theta.$$

考虑S^2上的光滑曲线 $\gamma(t)$, $t\in[a,b]$

其在球坐标下表示为 $(\varphi(t),\theta(t))$

则
$$L(\gamma):=\int_a^b\sqrt{\dot\varphi(t)^2+\cos^2\varphi(t)\,\dot\theta(t)^2}\,dt$$

一个观察：

$$L(\gamma)\geq\int_a^b|\dot\varphi(t)|\,dt\geq\left|\int_a^b\dot\varphi(t)\,dt\right|=|\varphi(b)-\varphi(a)|$$

且"="成立 $\Leftrightarrow$ $\dot\theta(t)=0$ 和 φ 单调.

$\Updownarrow$

$\theta(t)\equiv\text{const}.$

故两点之间 (a,b)（注意由于坐标的限制 a,b 不能是对径点）经线最短 □

回忆我们把两点之间的距离定义为连接它们的分段光滑曲线长度的下确界。那么自然的问题是两点之间是否一定存在长度最短的曲线？如果存在是否唯一？

为求最短线，我们考虑长度泛函的临界点。但长度泛函积分项带$\sqrt{\ }$，处理起来比较麻烦。但如下的观察说明，我们可以考虑能量泛函代替之。

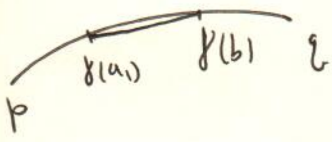
引理1. 对每条光滑曲线 $\gamma:[a,b]\to M$，我们有
$$L(\gamma)^2 \le 2(b-a)E(\gamma),$$
且等式成立当且仅当 $\sqrt{\langle\dot\gamma(t),\dot\gamma(t)\rangle_{\gamma(t)}} := \|\dot\gamma(t)\| \equiv \text{const}.$

证明: Hölder 不等式
$$L(\gamma) := \int_a^b \|\dot\gamma(t)\|\,dt \le \left(\int_a^b 1\,dt\right)^{1/2}\cdot\left(\int_a^b \|\dot\gamma(t)\|^2\,dt\right)^{1/2}$$
$$= (b-a)^{1/2}(2E(\gamma))^{1/2}.$$

"=" 成立当且仅当 $\|\dot\gamma(t)\| \equiv \text{const}.$ □

回忆曲线的长度不依赖于参数化的选取，故寻找最短线我们只须考虑弧长参数化的曲线，即 $\|\dot\gamma(t)\|=1$. 在此情形，
$$L(\gamma)^2 = 2\underbrace{(b-a)}_{L(\gamma)}E(\gamma) \Rightarrow L(\gamma) = 2E(\gamma)$$
故引理说明我们只须最小化 $E(\gamma)$.

下面我们用变分法来找如果存在最短线，它应该满足的方程。注意到如果 $\underset{:[a,b]\to M}{\gamma} \in C_{p,q}$ 是连接 p,q 的最短线，那么对
$$a \le a_1 < b_1 \le b,$$
γ 也必是连接 $\gamma(a_1)$ 和 $\gamma(b_1)$ 的最短线.



故我们可把问题局部化，不妨设 p,q 都含在一个坐标邻域 $(U, x^1,\dots,x^n)$ 之中。

引理2. 能量泛函的 Euler-Lagrange 方程为
$$(\ast\ast\ast)\qquad \ddot x^i(t) + \Gamma^i_{jk}(x(t))\,\dot x^j(t)\,\dot x^k(t) = 0,\quad i=1,\dots,n.$$

其中 $\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})$, $g_{jl,k} := \frac{\partial}{\partial x^k} g_{jl}$

称为 Christoffel symbols.

<u>定义1</u> (测地线) 满足下面方程的光滑曲线 $\gamma:[a,b]\to M$

$(\dot{x}^i(t) = \frac{d}{dt} x^i(\gamma(t)))$

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\,\dot{x}^j(t)\,\dot{x}^k(t) = 0, \quad i=1,2,\dots,n$$

称为<u>测地线</u>.

Christoffel 是一位德国数学家, 是一位开拓的大师 (陈省身). 他在柏林出生了, 曾在 ETH Zürich 工作. 最后在 Strasburg 大学工作 (在那时为普鲁士, 现属法国). 在黎曼 1854年演讲之后, 其讲稿直到1868年才发表. Christoffel 于1869年在 Crelle's Journal (Journal für ~~Re~~ die reine und angewandte Mathematik) 上发表文章, 讨论两个二次微分形式 (quadratic differential form)

$$F = \sum_{i,k} \omega_{ik}\, dx^i dx^k \qquad F' = \sum_{i,k} \omega'_{ik}\, dx'^i dx'^k$$

可<u>通过</u>独立变量变换 $\{x^i\} \to \{x'^i\}$ 把F变为F'的必要条件. 也就在这篇文章中, 他引入了 Christoffel 符号.

他的工作发表后, 影响意大利数学家 Gregorio Ricci-Curbastro 在 1883-1888年间发表六篇文章系统研究 Christoffel 的方法, 引入一种新的分析, 将 Christoffel's algorithm 解释成 "共变(协变)微分". Ricci (1893) 称之为 "absolute differential calculus"

不依赖于独立变量变换.

后来 1901年，Ricci 和他的学生 Levi-Civita 用法语将 Ricci's calculus 发表在 Klein's journal (Mathematische Annalen) 上. 后来被称为"张量分析"

后来广义相对论的发展促进了这些数学的发展。Einstein (1914) 在柏林演讲的第七节 "Geodesic line or equations of the point motion" 给出了用 Christoffel symbols 的不依赖于坐标选取的测地线方程

$$\frac{d^2 x_\tau}{ds^2} = \sum_{\mu\nu} \left\{ {\mu\nu \atop \tau} \right\} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}.$$

后来 Levi-Civita (1916/1917) 认识到 Christoffel symbols 的几何意义：决定向量的"平行移动"，这才又把 Christoffel, Ricci 这些远离了几何的讨论重新和几何联系起来。 Lecture 3. 2017.0

引理2的证明：记 $x(t) := (x^1(t), x^2(t), \ldots, x^n(t))$

先说明一般地一个泛函

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t))\, dt$$

的 Euler-Lagrange 方程为

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0, \quad i = 1, \ldots, d.$$

这是因为：设 $y(t) = (y^1(t), \ldots, y^n(t))$ 满足 $y(a) = y(b) = 0$.

考虑 $I(x + \varepsilon y)$

解 $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I(x + \varepsilon y) = 0$

我们有 $0 = \int_a^b \left(\frac{\partial f}{\partial x^i} y^i(t) + \frac{\partial f}{\partial \dot{x}^i} \dot{y}^i(t) \right) dt$

$$= \int_a^b \left(\frac{\partial f}{\partial x^i} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} \right) y^i(t)\, dt. \quad \forall y$$

由变分法基本引理，我们有

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0, \quad i = 1, \cdots, n.$$

它称为 I 的 E-L 方程.

特别地对能量泛函，我们有 E-L 方程为 $\left[\begin{array}{l} f(t, x(t), \dot{x}(t)) \\ = g_{jk}(x(t)) \dot{x}^j(t) \dot{x}^k(t) \end{array} \right.$

$$\frac{d}{dt}\left[g_{ik}(x(t)) \dot{x}^k(t) + g_{ji}(x(t)) \dot{x}^j(t) \right] - g_{jk,i}(x(t)) \dot{x}^j \dot{x}^k(t)$$

$$= 0, \quad i = 1, \cdots, n.$$

因此有

$$\underline{g_{ik,l} \dot{x}^l \dot{x}^k} + g_{ik} \ddot{x}^k + \underline{g_{ji,l} \dot{x}^l \dot{x}^j} + g_{ji} \ddot{x}^j - \underline{g_{jk,i} \dot{x}^j \dot{x}^k} = 0$$

$$i = 1, \cdots, n.$$

$$2 g_{im} \ddot{x}^m + (g_{ik,j} + g_{ji,k} - g_{jk,i}) \dot{x}^j \dot{x}^k = 0$$

$$i = 1, \cdots, n.$$

换指标 $i \to l$

$$(*) \quad 2 g_{lm} \ddot{x}^m + (g_{lk,j} + g_{jl,k} - g_{jk,l}) \dot{x}^j \dot{x}^k = 0, \quad l = 1, \cdots, n.$$

$\Rightarrow$ g^{il} $\forall i$, 加和 l, 有

$$g^{il} g_{lm} \ddot{x}^m + \frac{1}{2} g^{il} (g_{lk,j} + g_{jl,k} - g_{jk,l}) \dot{x}^j \dot{x}^k = 0$$

$$\Rightarrow \quad \ddot{x}^i + \frac{1}{2} g^{il} (g_{lk,j} + g_{jl,k} - g_{jk,l}) \dot{x}^j \dot{x}^k = 0$$

$$i = 1, \cdots, n. \qquad \square$$

<u>Remark 1</u>. When calculate the term $\Gamma^i_{jk} \dot{x}^j \dot{x}^k$, Notice that

$$\Gamma^i_{jk} \dot{x}^j \dot{x}^k = \frac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{lk,j}) \dot{x}^j \dot{x}^k$$
$$= \frac{1}{2} g^{il} (2 g_{jl,k} - g_{lk,j}) \dot{x}^j \dot{x}^k.$$

<u>Remark 2</u> As mentioned before, we only need to consider curves parametrized by <u>arc length</u> when looking for shortest curves. Now, we explain the other aspect: The solution of the equations (***) on page 26, i.e., every geodesic, is parametrized proportionally to arc length.

<u>Explanation</u>. $\frac{d}{dt}\left(g_{ij}(x(t))\,\dot{x}^i(t)\,\dot{x}^j(t)\right)$

$$= g_{ij}\ddot{x}^i\dot{x}^j + g_{ij}\dot{x}^i\ddot{x}^j + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k$$

$$= 2g_{ij}\ddot{x}^i\dot{x}^j + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k$$

change the indexes

$$= 2g_{lm}\ddot{x}^m\dot{x}^l + g_{lj,k}\dot{x}^l\dot{x}^j\dot{x}^k$$

(*) on page 29

$$\overset{\text{(*)}}{=} -(g_{lk,j} + g_{jl,k} - g_{jk,l})\dot{x}^l\dot{x}^j\dot{x}^k + g_{lj,k}\dot{x}^l\dot{x}^j\dot{x}^k$$

$$= (g_{jk,l} - g_{lk,j})\dot{x}^l\dot{x}^j\dot{x}^k$$

$$= 0$$

Hence $\langle \dot{x}, \dot{x}\rangle \equiv \text{const}$.

That is, every geodesic is parametrized proportionally to arc length.

<u>Remark 3</u>. (Curves in TM). We explain another viewpoint about the geodesic. First, any smooth curve ~~can be considered~~ in M gives a curve in its tangent bundle TM.

① System of coordinates. The total space of TM is the set of pairs (q, v), $q \in M$, $v \in T_qM$. Let $(U, x^1, \dots, x^n)$

be a coordinate neighborhood of M. $\forall q \in U$, any vector in T_qM can be written as

$$y^i \frac{\partial}{\partial x^i}$$

Recall locally we have $TU = U \times \mathbb{R}^n$. Then

$$(U \times \mathbb{R}^n, \ x^1, \cdots, x^n, y^1, \cdots, y^n)$$

is a coordinate neighborhood of $(q, v) \in TM$.

Then one can show that we obtain a differentiable structure for TM.

② ~~A~~ let $t \to \gamma(t)$ be a C^∞ curve in M.

then, it determines the curve $t \to (\gamma(t), \dot\gamma(t)) \in TM$.

If, moreover, γ is a geodesic in M, then the curve $t \to (\gamma(t), \dot\gamma(t))$ in terms of coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n)$

$$t \to (x^1(t), \cdots, x^n(t), y^1(t), \cdots, y^n(t))$$

satisfies
$$\begin{cases} \dot x^k(t) = y^k(t) \\ \dot y^k(t) + \Gamma^k_{ij}(x(t))\, y^i y^j = 0 \end{cases}, \ \forall k = 1, \cdots, n.$$

Local ~~uniqueness~~ Existence and uniqueness of geodesics

From ODE theory : (See Proposition 2.5 in do Carmo, Chapter 3 and discussions before that proposition)

Theorem 1 : For any $p \in M$, there exists

- an open set $V \subset M$, $p \in V$
- numbers $\delta > 0$ and $\varepsilon > 0$
- a C^∞ mapping : $\gamma : (-\varepsilon, \varepsilon) \times \mathcal{U} \to M$.
  $\mathcal{U} = \{(q, v) : q \in V, v \in T_qM, \|v\| < \delta\}$

such that, the curve $t \to \gamma(t, q, v)$, $t \in (-\varepsilon, \varepsilon)$

is the <u>unique</u> geodesic of M which satisfies

$$\gamma(0, q, v) = q, \quad \dot\gamma(0, q, v) = v.$$

Riemannian

for each $q \in V$, and each $v \in T_qM$ with $\|v\| < \delta$.

<u>Remark</u>: Let's have a closer look at the relations between the domain $(-\varepsilon, \varepsilon)$, and the length of the velocity $\|v\| < \delta$

Fix $q \in M$, let's denote $\gamma_v(t)$ as the geodesic with

$$\gamma_v(0) = q, \quad \dot\gamma_v(0) = v.$$

Then we claim $\quad \gamma_{\lambda v}(t) = \gamma_v(\lambda t)$, ~~$\forall \lambda \in \mathbb{R}$~~.

[This is because: in local coordinates, $\gamma_v(t)$ is written as

$$(x^1(t), \ldots, x^n(t))$$

They satisfy
$$\begin{cases} \ddot x^i(t) + \Gamma^i_{jk}(x(t))\dot x^j(t)\dot x^k(t) = 0 \\ (\dot x^1(t), \ldots, \dot x^n(t)) = v. \end{cases}$$

$$\Rightarrow \begin{cases} \ddot x^i(\lambda t) + \Gamma^i_{jk}(x(\lambda t))\dot x^j(\lambda t)\dot x^k(\lambda t) = \lambda^2(\ddot x^i + \Gamma^i_{jk}(x)\dot x^j \dot x^k)\big|_{\lambda t} = 0. \\ (\dot x(\lambda t), \ldots, \dot x^n(\lambda t)) = \lambda v \end{cases}$$

Hence $\gamma_{\lambda v}(t) = (x^1(\lambda t), \ldots, x^n(\lambda t)) = \gamma_v(\lambda t)$.

$\Rightarrow$ <u>Lemma</u>. If $\gamma(t, q, v)$ is defined for $t \in (-\varepsilon, \varepsilon)$, $\|v\| < \delta$.

then $\gamma(t, q, \lambda v)$ is defined for $t \in (-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda})$ and $\|\lambda v\| < \lambda\delta$

<u>Corollary</u>. Let $p \in M$, $v \in T_pM$. Then $\exists$ $\varepsilon > 0$ and a unique geodesic $\gamma : [0, \varepsilon] \to M$ with $\gamma(0) = p$, $\dot\gamma(0) = v$.

~~Moreover~~

Proof: ~~By theorem 1, Where~~ Assign $s = \frac{\delta/2}{\|v\|}$.

then $\|sv\| < \delta$.

By theorem 1, $\exists\ \varepsilon_0 > 0$, and a unique geodesic

$$\gamma_{sv} : [0, \varepsilon_0] \to M. \text{ with } \gamma_{sv}(0) = p,\ \dot{\gamma}_{sv}(0) = sv$$

Hence $\gamma_v(t) = \gamma_{sv}\left(\frac{t}{s}\right)$ is a geodesic defined on $[0, s\varepsilon_0]$

Pick $\varepsilon = s\varepsilon_0$, by the uniqueness of Thm 1, we show this Corollary. □

Exercise 1. Compute the geodesic equation of S^2 in sphererical coordinates.

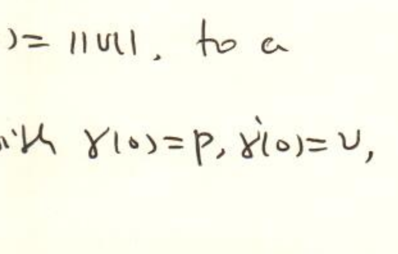
Exercise 2. What is the transformation behavior of the Christoffel symbols? Do they define a tensor?

§2. Minimizing Properties of Geodesics.

Next, we explain that a geodesic is "locally" shortest curve.

For that purpose, we first discuss the important concept

Exponential map.



Let (M, g) be a Rie mfld, $p \in M$.

Roughly speaking, the exponential map of M at p maps $v \in T_pM$, with $g_p(v,v) = \|v\|$, to a point q on the geodesic $\gamma_v : [0, b] \to M$ with $\gamma(0) = p,\ \dot{\gamma}(0) = v$, such that the arc length $\overline{pq} = \|v\|$.

This means, we should pick $q = \gamma_v(1)$. (Since as a geodesic, $\|\dot{\gamma}(t)\| = \|\dot{\gamma}(0)\| = \|v\|$.)

Definition 2 (Exponential Map) Let (M, g) be a Riemannian Mfld. $p \in M$. Denote

$$V_p := \{ v \in T_pM : \text{the geodesic } \gamma_v \text{ with } \gamma_v(0) = p, \dot{\gamma}_v(0) = v \text{ is defined on } [0,1] \}.$$

$$\exp_p : V_p \to M$$
$$v \longmapsto \gamma_v(1)$$

is called the exponential map of M at p.

In the following we use $\gamma_{p,v}$ to denote the geodesic with $\gamma_{p,v}(0) = p$, $\dot{\gamma}_{p,v}(0) = v$. (Oftenly, p is omitted)

What does V_p looks like?

① Star-shaped around $0 \in T_pM$.

If $v \in V_p$, i.e. γ_v is defined on $[0,1]$, then $\gamma_{\lambda v}$ $0<\lambda<1$ is defined on $[0, \frac{1}{\lambda}]$, and, in particular, on $[0,1]$. Hence $v \in V_p \Rightarrow \lambda v, 0<\lambda\leq 1 \in V_p$.

②. $\forall p \in M$, $\exists\, \varepsilon_0$ s.t. $B(0, \varepsilon_0) \subset V_p$

i.e. $\forall\, w \in T_pM, \|w\| \leq \varepsilon_0$, we have $\gamma_{p,w}$ is defined on $[0,1]$.

By Thm 1, $\exists\, \varepsilon, \delta > 0$ $\forall\, v \in T_pM, \|v\| < \delta$, $\gamma_{p,v}$ is defined on $[0, \varepsilon]$

hence $\gamma_{p,\varepsilon v}$ is defined on $[0,1]$.

$\Rightarrow \forall\, w \in T_pM$ with $\|w\| \leq \varepsilon\|v\| < \varepsilon\delta$,

we have $\gamma_{p,w}$ is defined on $[0,1]$

That is, $w \in V_p$. □

~~$\{v \in T_pM \mid \|v\|=1\}$ is compact, we see there exists $\varepsilon_0>0$, s.t. $\forall\, v\in T_pM$, $\|v\|=1$, γ_v is defined on $[0,\varepsilon_0]$. By the argument in ①, we can use $\gamma_{\varepsilon_0 v}$ is defined on $[0,1]$~~

<u>Example</u>. (i) $M=\mathbb{R}^n$, $g_{ij}=\delta_{ij}$

the geodesic equation. $\ddot{x}^i(t)=0$.

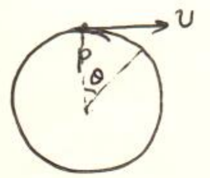
the geodesics are straight lines parametrized proportionally to arc length.

$\forall\, p\in\mathbb{R}^n$, $v\in T_p\mathbb{R}^n$ ← identify it with $\mathbb{R}^n$.

$\exp_p(v) = p+v$.

$V_p = T_p\mathbb{R}^n=\mathbb{R}^n$

(ii). Circle $(S^1, d\theta\otimes d\theta)$



$p\in S^1$, T_pS^1 can be identified with $\mathbb{R}$.

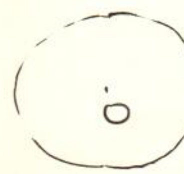
then $\exp_p(v)=e^{iv}$ $\quad(p=e^{i0}=1)$.

$V_p=T_pS^1=\mathbb{R}$. (In local coordinates, $\exp_p: v\mapsto v$.)

This is the simplest example explaining why the terminology "exponential map" is used. It actually comes from Lie theory.

(iii) Open disc in $\mathbb{R}^2$: $D_0=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$

with a Riemannian metric induced from the canonical Euclidean metric on $\mathbb{R}^2$. $\exp_0(v)=0+v=v$



But $V_0\neq\mathbb{R}^2$, $V_0=D_0$ (we identify T_0D_0 with $\mathbb{R}^2$)

(35)

Thm 1. The exponential map $\exp_p$ maps a neighborhood of $0 \in T_pM$ <u>diffeomorphically</u> onto a neighborhood of $p \in M$.

~~Proof~~ <u>Remark</u>: ~~We~~ Reason for restricting to a neighborhood:

① $\exp_p$ may not be defined on the whole T_pM.

② even if $\exp_p$ is defined on the whole T_pM, it ~~still~~ ~~came~~ is not necessarilly a diffeomorphism.

(Example of $(S^1, d\theta \otimes d\theta)$, $\exp_p$ is not injective)

<u>Proof</u>: ~~dexp~~ $0 \in T_pM$.

$$d\exp_p(0) : T_0(T_pM) \to T_pM.$$

Since T_pM is a vector space, we may identify T_0T_pM with T_pM.

$$\Rightarrow \quad d\exp_p(0) : T_pM \to T_pM.$$

Now we calculate $d\exp_p(0)(v)$ for a $v \in T_0(T_pM) = T_pM$.

<u>Recall</u>: for a C^∞ map $f: M \to N$, $x \mapsto y$.

One way to calculate $df: T_xM \to T_yN$ is the following:

For any $v \in T_xM$, consider a curve γ with $\gamma(0) = x$, $\dot\gamma(0) = v$.

Then $\xi = f(\gamma)$ is a curve in N, and

$$df(v) = \dot\xi(0)$$

Recall $Xf_{(p)} = \frac{d}{dt} f(\gamma(t))|_{t=0}$ where $\gamma(0)=p$, $\dot\gamma(0) = X$.

Here, $\exp_p$: ~~T_pM~~ $V_p \subset T_pM \to M$, $0 \mapsto p$.

For $v \in T_0(T_pM) = T_pM$, consider $\gamma(t) = tv$

we have

$$d\exp_p(0)(v) = \frac{d}{dt}\exp_p(tv)\Big|_{t=0}$$

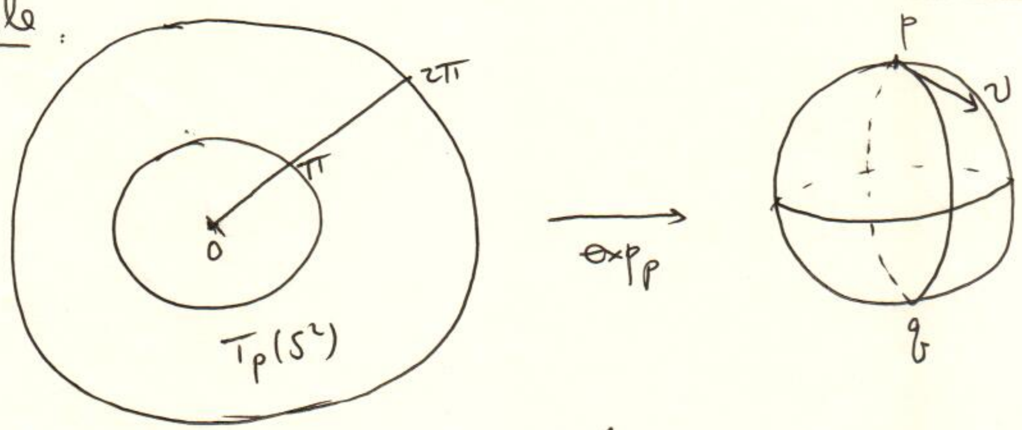
$$= \frac{d}{dt}\gamma_{tv}(1)\Big|_{t=0} = \frac{d}{dt}\gamma_v(t)\Big|_{t=0} = \dot\gamma_v(0) = v.$$

That is $d\exp_p(0) = id_{|T_pM}$.

In particular, $d\exp_p(0)$ has maximal rank, and by "inverse function theorem", there exists <u>a neighborhood</u> of $0 \in T_pM$, which is mapped diffeomorphically onto a neighborhood of $p \in M$. $\square$

Lecture 4, 2017.03.02

<u>Example</u>:



$S^2 \subset \mathbb{R}^3$. q is the antipodal point of p.

$\exp_p$ is defined on the whole T_pS^2.

Let $B(0,\pi) \subset T_p(S^2)$ be the open ball around 0 in $T_p(S^2)$
(with the scalar product given by the Rie. metric of S^2)

$\exp_p : B(0,\pi) \to S^2 \setminus \{q\}$ ~~injectively, and hence~~, diffeomorphically.

$\overline{B(0,\pi)} \setminus B(0,\pi) = \partial B(0,\pi) \to \{q\}$

open anulus $B(0,2\pi) \setminus \overline{B(0,\pi)} \to S^2 \setminus \{p,q\}$

$\overline{B(0,2\pi)} \setminus B(0,2\pi) \to \{p\}$.

And we can identify T_pM with $\mathbb{R}^n$ via

$$\Phi : T_pM \to \mathbb{R}^n$$
$$v = v^i e_i \longmapsto (v^1, \cdots, v^n).$$

Thm 2. (page 36) tells that $\exists$ a neighborhood $U \ni p$, s.t. $\exp_p^{-1}$ map U diffeomorphically onto a neighborhood of $0 \in T_pM \overset{\text{via } \phi}{=} \mathbb{R}^n$. In particular, $p \mapsto 0$.

Definition (Normal coordinates) The local coordinates defined by ~~the~~ $(U, \exp_p^{-1})$ are called (Riemanian) **normal coordinates** with center p.

The advantage of such a choice of coordinates is ~~so~~ presented in the following result:

Theorem 3 . In normal coordinates, we have for the Riemannian metric, and all i, j, k.

$$g_{ij}(0) = \delta_{ij} \qquad (1)$$
$$\Gamma^i_{jk}(0) = 0, \text{ (and also } g_{ij,k}(0) = 0) \qquad (2)$$

Proof: (1) follows from the identification ϕ of T_pM and $\mathbb{R}^n$.

(Recall $g_p(e_i, e_j) = \delta_{ij}$).

Next, we show (2), Recall in the local coordinate $(U, \exp_p^{-1}) = (U, v^1, \cdots, v^n)$, the geodesic equation is

$$\ddot{v}^i + \Gamma^i_{jk}(t)\, \dot{v}^j(t)\, \dot{v}^k(t) = 0, \quad i = 1, 2, \cdots, n. \qquad (4)$$

On the other hand, in $\exp_p^{-1}(U) \subset \mathbb{R}^d$, the line tv, $t \in \mathbb{R}$, $v \in \mathbb{R}^d$ is $\exp_p^{-1}(\gamma_{tv}(1)) = \exp_p^{-1}(\gamma_v(t))$, i.e. is the image of a geodesic in M via the coordinate map.

<u>Remark</u>: Even if $\exp_p$ is defined on the whole T_pM, it may be not a global diffeomorphism. Suppose

$$\exp_p : B(0,\rho) \to \exp_p(B(0,\rho))$$

is diffeomorphic, how large can ρ be?

Here we mention the following concepts of <u>injectivity radius</u>.

<u>Definition</u>: Let M be a Rie. mfld, $p \in M$. The injectivity radius of p is

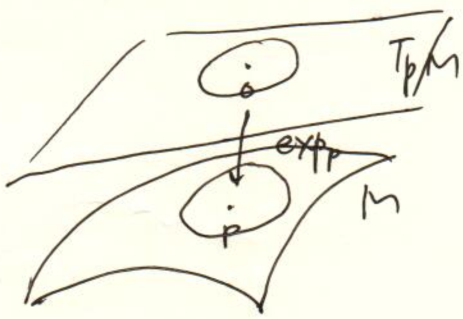
$$i(p) := \sup\{\rho > 0 : \exp_p \text{ is a diffeomorphism on } B(0,\rho) \subset T_pM\}.$$

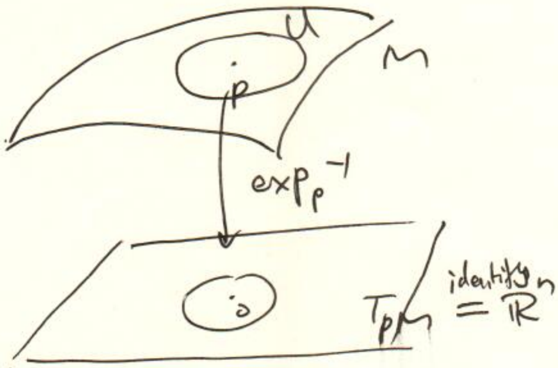
The injectivity radius of M is

$$i(M) := \inf_{p\in M} i(p).$$

The above example shows that $i(S^2) = \pi$.

Normal coordinates





T_pM has an ~~scalar~~ inner product defined by g.

Let $e_1, \dots, e_n$ $(n = \dim M)$ be an orthonormal basis of T_pM (w.r.t. the inner product given by g). Then for each $v \in T_pM$, we can write

$$v = v^i e_i$$

Therefore, $v(t) = tv$ satisfies $(\ast)$

This implies $\Gamma^i_{jk}(tv)\, v^j v^k = 0, \quad i=1,\dots,n, \ \forall\, v \in \mathbb{R}^d$.

↳ dependent on v.

In particular, for $t=0$

$$\Gamma^i_{jk}(0)\, v^j v^k = 0, \quad i=1,2\dots,n, \ \forall\, v \in \mathbb{R}^d. \quad (\ast\ast)$$

For any indices l and m, pick $v = e_l + e_m$, ~~i.e. of~~ we have

$$\Gamma^i_{lm}(0) = 0, \quad i=1,2,\dots,n.$$

That is $\boxed{\Gamma^i_{jk}(0) = 0, \ \forall\, i,j,k}$.

Recall the definition of $\Gamma^i_{jk}(0)$, we obtain

at $0 \in \mathbb{R}^d$: $g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l}) = 0, \ \forall\, i,j,k$.

$\Rightarrow \quad g_{jl,k} + g_{lk,j} - g_{jk,l} = 0, \ \forall\, j,k,l$.

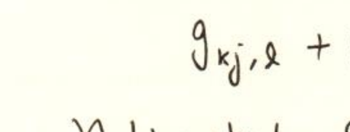
By a cyclic permutation of the indices ~~$j \to k,\ k \to$~~ $\begin{matrix} j & k & l \\ \downarrow & \downarrow & \downarrow \\ k & l & j \end{matrix}$

$$g_{kj,l} + g_{jl,k} - g_{kl,j} = 0. \quad \cancel{\forall j,k,l}.$$

Noting that $g_{lk,j} = g_{kl,j}$, $g_{jk,l} = g_{kj,l}$, we got

$$2 g_{jl,k} = 0 \quad \text{at } 0 \in \mathbb{R}^d. \qquad \square$$

<u>Remark</u>: In general, the second derivatives of the metric cannot be made to vanish at a given point by a suitable choice of local coordinates. The obstruction is given by the "curvature tensor".



$T_pM \overset{\Phi}{\cong} \mathbb{R}^n$

on $\mathbb{R}^n$ we can introduce the standard polar coordinates

$(r, \varphi^1, \dots, \varphi^{n-1})$

where $\varphi = (\varphi^1,\dots,\varphi^{n-1})$ parametrizes the unit sphere S^{n-1}.

Then via Φ, we obtain polar coordinate on T_pM. (44)

We can write the metric g in polar coordinate:

$$\begin{array}{ccc} g_{rr} = g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}), & g_{r\varphi} & g_{\varphi\varphi} \\ \shortparallel & \shortparallel & \shortparallel \\ g_{11} & g_{1\ell},\ \ell = 2, \dots, n & (g_{k\ell})\ k, \ell = 2, \dots, n. \end{array}$$

In particular at $0 \in T_pM$, we have

$$g_{rr}(0) = 1, \quad g_{r\varphi}(0) = 0. \qquad (*)$$

(The same reason as (1) in Thm 3 on page (38))

The point is that in this case we can show $(*)$ holds true <u>not only</u> at $0 \in T_p(M)$ $(= \mathbb{R}^n)$, but in the whole coordinate neighborhood.

<u>Thm 4</u>: For the polar coordinates, obtained by transforming the Euclidean coordinates of $\mathbb{R}^d$, on which the normal coordinates with centre p are based, into polar coordinates, we have

$$g_{ij} = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & g_{\varphi\varphi}(r, \varphi) \\ 0 & \end{pmatrix}$$

when $g_{\varphi\varphi}(r, \varphi)$ is the $(n-1) \times (n-1)$ matrix of the components of the metric w.r.t. angular variables $(\varphi^1, \dots, \varphi^{n-1}) \in S^{n-1}$.

<u>Proof</u>: In this case, tv, $v \in \mathbb{R}^n$ is transformed to be $\varphi \equiv$ const. That is, $\varphi \equiv$ const are geodesic when parametrized by arc length in the local coordinates. They are given by

$$x(t) = (t, \varphi_0). \quad \varphi_0 \text{ fixed}.$$

Geodesic equation gives

$$\Gamma^i_{rr}(x(t)) \cdot \dot{t}\dot{t} = \Gamma^i_{rr}(x(t)) = 0, \forall i \qquad (\dot{\varphi}_0 = 0).$$

Compare with the situation in (**) on page (40).

Hence at $x(t) \in T_pM = \mathbb{R}^n$,

$$g^{il}\,(\underbrace{g_{rl,r} + g_{lr,r}}_{2g_{rl,r}} - g_{rr,l}) = 0, \ \forall i$$

$\Rightarrow$

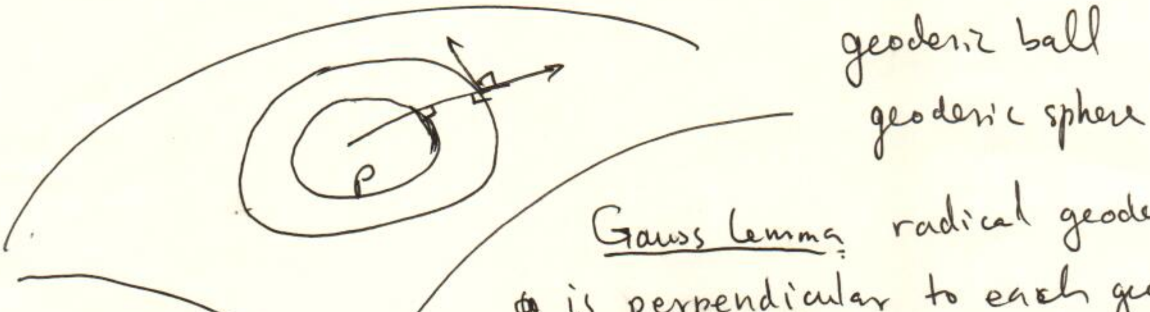
$$2g_{rm,r} - g_{rr,m} = 0, \ \forall\ m. \qquad (\maltese\,\maltese)$$

For $m = r$, $\Rightarrow$ $g_{rr,r} = 0$.

combining with $g_{rr}(0) = 1$ $\Rightarrow$ $g_{rr} \equiv 1$.

~~Pick m =~~ Hence $g_{rr,m} = 0, \forall\ m$, $(\maltese\,\maltese) \Rightarrow g_{r\varphi,r} = 0, \forall \varphi$.

$\Rightarrow$ $g_{r\varphi} \equiv 0$. $\square$



Gauss Lemma: radical geodesic is perpendicular to each geodesic sphere.

Remark: $dr \otimes dr + g_{\varphi\varphi}(r,\varphi)\, d\varphi \otimes d\varphi$

is not a product metric, since $g_{\varphi\varphi}$ may depend on r.

Remark: Since g is positive definite, we have

$$(g_{\varphi\varphi}(r,\varphi))$$

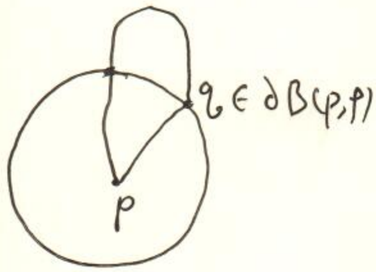
is also positive definite.

Corollary 1. (i) For any $p \in M$, $\exists\ \rho > 0$ s.t. the (Riemannian) polar coordinates may be introduced on $\overline{B(p,\rho)} := \{q \in M, d(p,q) \leq \rho\}$

(ii) For any such ρ, and $q \in \partial B(p,\rho)$, there is a unique normal geodesic whose length $(=\rho)$ is the shortest one among all curves $\in C_{p,q}$.

Proof: (i). By Thm 2, polar coord. can be introduced (43)

~~by~~ on a neighborhood U of p. Since mfld top. and metric top. coincides, such a ρ can be found.



(ii) Consider any ~~c~~ $c \in C_{p,q}$. W.o.l.g., we assume ~~c~~ c smooth. ~~In polar coord.,~~

~~we write~~ $c : [0,T] \to M$

~~$c(t) = (r(t), \varphi(t)) : 0 \le t \le T$~~

~~c(t) m~~ c may ~~not~~ leave our polar coord. neighborhood.

Let $t_0 := \inf\{t \le T : d(c(t), p) \ge \rho\}$

Then $c|_{[0,t_0]} \subset \overline{B(p,\rho)}$.

In polar coord., write

$$c(t) = (r(t), \varphi(t)), \quad c(t_0) = (\rho, \varphi(t_0))$$

We calculate

$$L(c_{|[0,t_0]}) = \int_0^{t_0} \sqrt{g_{ij}(c(t))\dot{c}^i(t)\dot{c}^j(t)}\, dt$$

$$= \int_0^{t_0} \sqrt{g_{rr}(c(t))\dot{r}\dot{r} + \underbrace{g_{\varphi\varphi}(c(t))\dot{\varphi}\dot{\varphi}}_{\ge 0}}\, dt$$

$$\overset{g_{rr}(c(t))=1}{\ge} \int_0^{t_0} |\dot{r}|\, dt \ge \left| \int_0^{t_0} \dot{r}\, dt \right| = |r(t_0) - r(0)| = \rho.$$

Moreover, "=" holds iff. $g_{\varphi\varphi}\dot{\varphi}\dot{\varphi} \equiv 0$ and $\dot{r} \ge 0$ or $\dot{r} \le 0$.

$$\Updownarrow$$

$$\dot{\varphi} = 0 \Rightarrow \varphi \equiv \text{const}.$$

Hence the "=" holds iff.

$$c(t) = (t, \varphi_0), \quad \text{when } q = (\rho, \varphi_0).$$

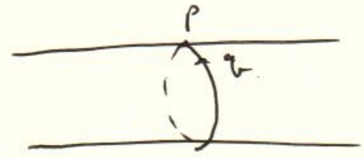
Recall (t, φ_0) is the geodesic in the polar coordinates. □

Remark: (1) From the proof, we ~~say~~ see $\forall\, c \in C_{p,q}$,

$L(c) \ge L(\gamma)$, where γ is the ~~unique~~ radical geodesic.

and "=" iff c is a monotone reparametrization of γ. (4)

(2). There may exists other geodesics from p to q whose length is longer.



That is the "shortestness property of a geodesic" is not global !!!

From Corollary 1, we see $\forall p, q \in M$, when they are close "enough" to each other, then there exists precisely one geodesic of shortest length. Can we have a uniform description of the "closedness" which ensure the existence of shortest geodesics, at least when M is cpt? For this purpose, we first discuss a refinement of Thm 2 (page 36) ($d\exp_p(0) = I$) and the "<u>totally normal neighborhood</u>".

<u>Theorem 5</u>. (totally normal neighborhood). For any point $p \in M$, there exists a neighborhood W of p and a number $\delta > 0$, such that, for every $q \in W$, $\exp_q$ is a diffeomorphism on $B(0,\delta) \subset T_qM$, (in other words, injectivity radius $i(q) \geq \delta$), and $\exp_q(B(0,\delta)) \supset W$.

<u>Remark 1</u>. (Terminologies) If $\exp_p$ is a diffeomorphism of a neighborhood V of the origin in T_pM. Then we

we call $\exp_p(V) =: U$ a <u>normal neighborhood</u> of p.

Theorem 4 tells, $\exists$ a neighborhood W of p such that W is a normal neighborhood of each q of W. W is then called a <u>totally normal neighborhood</u> of $p \in M$.

If $B(0, \varepsilon)$ is such that $\overline{B(0,\varepsilon)} \subset V$, we call $\exp_p(B(0,\varepsilon))$ the <u>normal ball</u> with center p and radius ε. The geodesics in $\exp_p(B(0,\varepsilon))$ that begins at p are referred to as <u>radial geodesics</u>.

<u>Remark 2</u>: By <u>Corollary 1</u>, any 2 points in W can be connected by a unique minimizing geodesic.

<u>Proof</u>: For the proof, we first discuss a refinement of <u>Thm 2</u> (page 36). There exists a neighborhood V of p and $\delta > 0$,

$$\mathcal{U} := \{(q, v) \in TM : q \in V, v \in T_qM, \|v\| < \delta\}$$

such that $\exp : \mathcal{U} \to M$ is well defined.

$$(q,v) \mapsto \exp_q v$$

Consider the following map.

$$F : \mathcal{U} \longrightarrow M \times M$$
$$(q,v) \longmapsto (q, \exp_q v)$$

In particular, we see $F(p,0) = (p,p)$

Then $dF_{(p,0)} : T_{(p,0)}(TM) \to T_{(p,p)}(M \times M)$.

<u>Lemma</u> For each $p \in M$ and with it the zero vector

$0 \in T_pM$, $dF(p,0)$ is nonsingular.

Proof: First note that, we can identify the tangent space

$$T_{(p,p)}(M\times M) \text{ to } T_pM \times T_pM.$$

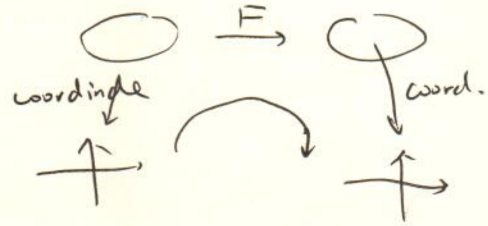
$$T_{(p,0)}(TM) \text{ to } T_pM \times T_0(T_pM) \cong T_pM\times T_pM.$$

$$F : \mathcal{U} \to M\times M$$

In local coordinates, this map can be considered as

$$(x^1,\dots,x^n, v^1,\dots,v^n) \to (x_1^1,\dots,x_1^n, x_2^1,\dots,x_2^n)$$

We consider $dF_{(p,0)}$ as a linear map



$$T_pM\times T_pM \to T_pM\times T_pM,$$

~~then on the first factor to the first factor; the map~~

- varying p. F is identity in the first coordinate. Hence ~~this~~ on the first factor to the first factor $dF_{(0,p)}$ is identity.
- fix p and vary v in T_pM, the first coordinate of F is fixed and the second coordinate is $\exp_p v$.

Hence, $dF_{(0,p)}$ is identically 0 from the second factor to the first and identity from the second factor to the second factor. (Thm 20, page 36)

$$\begin{pmatrix} dF^1_{v=0} & dF^1_{q=p} \\ dF^2_{v=0} & dF^2_{q=p} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$

$\square$.