Proof of Theorem F :

By Lemma, and the inverse function theorem we know that F is a local diffeomorphism.

This means that $\exists$ a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of $(p,0) \in TM$ s.t. F maps $\mathcal{U}'$ diffeomorphically onto a neighborhood $\underset{\sim}{W}'$ of $(p,p) \in M \times M$.

By shrinking $\mathcal{U}'$ if necessary, we can take $\mathcal{U}'$ to be the form

$$\mathcal{U}' = \{(q,v) : q \in V', v \in T_q M, \|v\| < \delta\}$$

where $V' \subset V$ is a neighborhood of p in M.

Now choose a neighborhood W of p in M so that

$$W \times W \subset \underset{\sim}{W}'$$

Then from the definition of F, we see

$$\exp_p(B(0,\delta)) \supset W. \qquad \square$$

Now, we have some immediate consequence.

Corollary 2. Let Ω be a compact subset of a Rie. mfld M. There there exists $\rho_0 > 0$ with the property that for any $p \in \Omega$, Rie. polar coordinates may be introduced on $B(p,\rho_0)$

Proof. $\forall p \in \Omega$, we can find a totally normal neighborhood W_p of p. By compactness, we have a finite subcover $\{W_{p_i}\}_{i=1}^N$ of $\{W_p\}_{p\in\Omega}$ of Ω. Since for each W_p, $\exists$ a ρ_p s.t. Rie. polar coord. may be defined on $B(q,\rho_p)$ $\forall q \in W_p$.

We pick $\rho_0 = \min_{i=1,\dots,N}\{\delta_{p_i}\}$. $\square$

Corollary 3: Let Ω be a cpt subset of a Rie mfld M. Then there exists $\rho_0 > 0$ with the property that:

for any two points $p, q \in M$ with $d(p,q) \le \rho_0$

can be connected by precisely one geodesic of shortest path.

~~The geodesic depends continuously on p and q.~~

The geodesic depends continuously on (p,q). ~~in the following sense~~:

Proof. ρ_0 from Corollary 2. satisfies the first claim by Corollary 1. ~~The shortest geodesic from p to~~ Moreover, given (p,q), $d(p,q) \le \rho_0$, there exists a unique $v \in T_pM$ (given by $F^{-1}(p,q) = (p,v)$) that depends continuously on (p,q) ~~$q \in B(p,\rho_0)$ depends continuously on p~~ and is s.t. $\dot\gamma(0) = v$. $\square$

Corollary 4 Let M be a compact Rie. mfld, $i(M) > 0$.

$\boxed{\text{Local isometries map geodesics to be geodesics}}$

[JJ, §1.4]

Recall that a differentiable (C^∞) map $h: M \to N$ is a local isometry, if $\forall\, p \in M$, $\exists$ a neighborhood U for which

$h|_U : U \to h(U)$ is an isometry diffeomorphism and $h(U)$ open in N

and $g|_U = h^*(\gamma|_{h(U)})$

where $(g_{ij}(p))$, $(\gamma_{\alpha\beta}(h(p)))$ are the metrics on U, $h(U)$ resp.

In fact, $g_{ij}(p) = \gamma_{\alpha\beta}(h(p)) \dfrac{\partial h^\alpha(p)}{\partial x^i} \dfrac{\partial h^\beta(p)}{\partial x^j}$.

A local isometry has the same effect as a coordinate change. We have already see in the Homework Exercise 2, that the geodesic equations

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \left(\ddot{y}^\alpha + \tilde{\Gamma}^\alpha_{\eta\gamma}\dot{y}^\eta\dot{y}^\gamma\right)\frac{\partial x^i}{\partial y^\alpha}.$$

Hence geodesic is mapped to be geodesic,

(det$\left(\frac{\partial x^{\mu}}{\partial y^{\alpha}}\right) \neq 0$.)

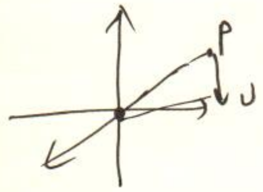
Intuitively, Isometries leave the lengths of tangent vectors and therefore also the lengths and energies of curves invariant. Thus, critical points, i.e. geodesics, are mapped geodesics.

This observation has interesting consequences.

<u>Example 1.</u> (geodesics of S^n).

The orthogonal group $O(n+1)$ operates isometrically on $\mathbb{R}^{n+1}$, and since it maps S^n into S^n, it also operates isometrically on S^n.

Let now $p \in S^n$, $v \in T_p(S^n)$



Let E be the two dimensional plane through the origin of $\mathbb{R}^{n+1}$, containing v.

<u>Claim</u>: the geodesic γ_v pass through p with tangent vector v is <u>the great circle through p with tangent vector v</u> (parametrized proportionally to arc length), i.e. the intersection of S^n and E.

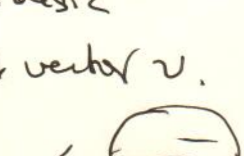
<u>Proof</u>: let $S \in O(n+1)$ be the reflection across E.

then $Sv = v$, $Sp = p$.

γ_v is geodesic $\Rightarrow$ $S\gamma_v$ is also a geodesic through p with tangent vector v.

by uniquess result, $\gamma_v = S\gamma_v$

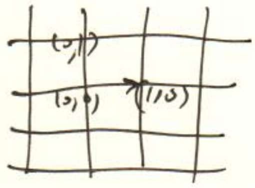
Hence image of γ_v is the great circle



$\square$

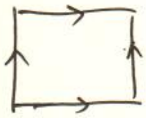
Examp 2 (geodesics on $\mathbb{T}^2$)

the covering map $\pi : \mathbb{R}^2 \to \mathbb{T}^2$



$\omega_1 = (1,0) \in \mathbb{R}^2$

$\omega_2 = (0,1) \in \mathbb{R}^2$.



Consider $z_1, z_2 \in \mathbb{R}^4$ as equivalent if $\exists\ m_1, m_2 \in \mathbb{Z}$ s.t.

$$z_1 - z_2 = m_1\omega_1 + m_2\omega_2.$$

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$$

$$z \longmapsto [z]$$

↳ equivalent classes.

differentiable structure:

$\Delta_\alpha \subset \mathbb{R}^2$ - open and does not contain equivalent pts.

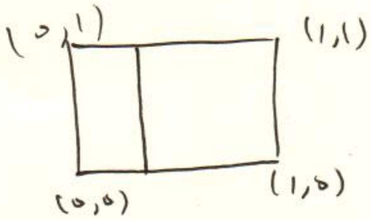
then $\left(U_\alpha := \pi(\Delta_\alpha),\ z_\alpha = (\pi|_{\Delta_\alpha})^{-1}\right)$.

For each chart $(U, (\pi|_U)^{-1})$ we use the Euclidean metric on $\pi^{-1}(U)$. Since the translations

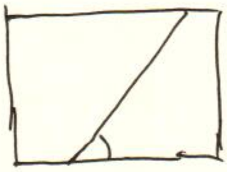
$$z \mapsto z + m_1\omega_1 + m_2\omega_2,\ m_1, m_2 \in \mathbb{Z}$$

are Euclidean isometries, the Euclidean metrics on the different components of $\pi^{-1}(U)$ (which are obtained from each other by translations.) yield the same metric on U. Hence the Riemannian metric on $\mathbb{T}^2$ is well defined, and $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is a local isometry.

Therefore, Euclidean geodesics of $\mathbb{R}^2$ are mapped onto geodesics of $\mathbb{T}^2$.



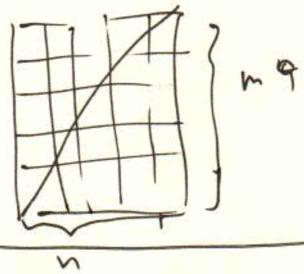
closed geodesic of length 1.



if the slope is rational, closed geodesics

if the slope is irrational, the image lies dense in T^2.

slope $= \frac{m}{n}$



[WSY, pp. 48-54]

[JJ, §1.6]

§3 Global Properties; Hopf-Rinow Theorem

In the last section, we know when two pts $p, q \in M$ are close enough to each other, there exists precisely one geodesic with the shortest length. Naturally, one would ask the following questions.

Question 1. If a curve γ is of shortest length, Is γ a geodesic?

Question 2. Let $\gamma : [0,1] \to M$ be a geodesic, is it the shortest curve from $\gamma(0)$ to $\gamma(1)$?

Question 3. Given $p, q \in M$, does there exists a curve from p to q with the shortest ~~curve~~ length?

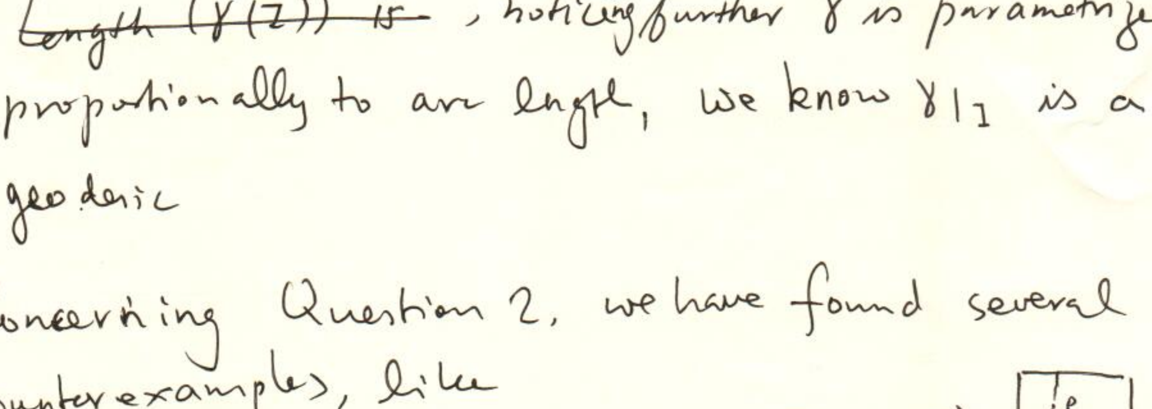
Recall that a geodesic γ has to be parametrized proportionally to its arc length. Hence, a proper way to formulate Question 1 is:

Question 1': Let $\gamma : [0, c] \to M$, $|\dot{\gamma}| = 1$
and $\forall\, \xi \in C_{p,q}$ (piecewise C^∞ curve from p to q),
Length$(\gamma) \le$ Length(ξ).
Is γ necessarily a geodesic?

Proposition 1. If a piecewise ~~different~~ C^∞ $\gamma : [a,b] \to M$ with parameter proportional to arc length, has length less than or equal to the length of any other piecewise C^∞ curve from $\gamma(a)$ to $\gamma(b)$, then γ is a geodesic. [In particular, γ is ~~regular~~ C^∞.]

Proof: Let $t \in [a,b]$, and let W be a totally normal neighborhood of $\gamma(t)$. There exists a closed interval $I \subset [a,b]$ with nonempty interior, $t \in I$ s.t. $\gamma(I) \subset W$. By the gloal "shortestness" property of γ, we know $\gamma(I)$ is a piece wise smooth curve ~~from~~ connecting two pts in W with the shortest length. By Corollary 4.11 (p.42) ~~Length $(\gamma(I))$ is~~, noticing further γ is parametrized proportionally to arc length, we know $\gamma|_I$ is a geodesic □

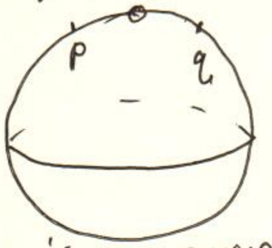
Concerning Question 2, we have found several counterexamples, like



So the answer to Question 2 is "NO!" (53)

Then one may ask when is a geodesic γ also a minimizing curve? We will discuss this issue in later lectures.

The answer to Question 3 is also "NO!". If a curve γ from p to q is of the shortest curve after choosing the parameter proportional to arc length, Proposition tells that γ must be a geodesic.



$= S^2 \setminus \{\text{north pole}\}$

then there is no curve from p to q with the shortest length. (but you have a minimizing sequence of curves).

When is the answer to Question 3 "Yes"?
It turns out, one have to require ~~that~~ M to be <u>complete</u> !!

Given a Riemannian mfld, (M,g), ~~we have~~ Recall that (M,g) with the distance function d derived from g is a metric space (M,d). And the topology of (M,d) coincides with the original topology of M. Therefore (M,d) is a complete metric space <u>iff</u> M is complete as a topological space w.r.t. its original topology. So we do not need to distinguish this two <u>completeness</u>.

Hopf-Rinow thm ~~will~~ tells completeness implies the existence of minimizing geodesic between any two points. Moreover, H-R also gives several equivalent description of completeness.

Theorem (Hopf – Rinow 1931)

(Über den Begriff der vollständigen differentialgeometrischen Fläche, Commentarii Mathematici Helvetici. (1931)) ← 瑞士数学会的杂志，1929年创办 Comment. Math. Helv.

Let M be a Riemannian mfld. The following statements are equivalent:

- (i) M is a complete metric space.
- (ii) The closed and bounded subsets of M is compact.
- (iii) $\exists\, p \in M$ for which $\exp_p$ is defined on all of T_pM.
- (iv) $\forall\, p \in M$, $\exp_p$ is defined on all of T_pM.

Each of the statements (i) – (iv) implies

(v) Any two points $p, q \in M$ can be joined by a geodesic of length $d(p,q)$, i.e. by a geodesic of shortest length.

<u>Digest of the theorem</u>

(1) (i) $\Rightarrow$ (v) but not vice versa.
Counterexample. open disc is not complete, but satisfies property (v).

(2) <u>Definition</u> (geodesically complete) A Rie. Mfld M is <u>geodesically complete</u> if for all $p \in M$, $\exp_p$ is defined on all of T_pM, or, in other words, if any geodesic $\gamma(t)$ with $\gamma(0) = p$ is defined for all $t \in \mathbb{R}$.

H-R Thm tells completeness $\Leftrightarrow$ <u>geodesical completeness</u>.

manifold topology
apriori ~~not~~ independent of the metric

depends on the Rie. metric

(3). (M, g) as a complete metric space is a very special one. as shown in (ii).

Consider a ~~discrete~~ countable set

$$\mathcal{S} = \{a_i : i = 1, 2, \cdots\}$$

with a discrete metric, i.e. $d : \mathcal{S} \times \mathcal{S} \to [0, \infty)$

s.t. $d(a_i, a_j) = \delta_{ij}$.

Then $(\mathcal{S}, d)$ is complete. and ~~~~ bounded.

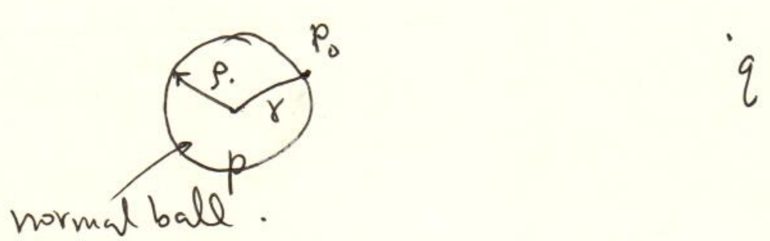
But $\mathcal{S}$ is not compact.

(4) <u>Corollary</u>: Let (M, g) be a complete Riemannian Manifold. Then $\exp_p : T_pM \to M$ is surjective for any $p \in M$.

<u>Proof of the theorem</u>:

The "core" result is (iv) $\Rightarrow$ (v):

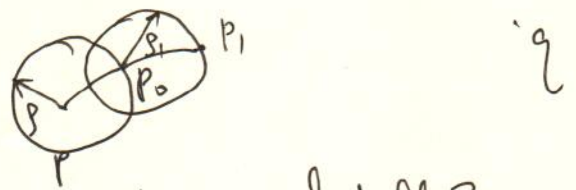
Given $p, q \in M$, we hope to find a shortest geodesic γ from p to q. ~~First~~ We know $\gamma(0) = p$, but how to decide $\dot\gamma(0)$?



Consider a normal ball $B(p, \rho)$. Since $\partial B(p, \rho)$ is compact, and $d(q, \cdot)$ is a continuous function, ~~~~

~~can~~ there exists $p_0 \in \partial B(p,\rho)$ s.t. $d(q,\cdot)$ attains its minimum on $\partial B(p,\rho)$ at p_0.

Now the idea is the following:



At p_0, consider the normal ball $B(p_0, \rho_1)$, find p_1 to be the point at which $d(q,\cdot)$ attains its minimum on $\partial(B(p_0,\rho_1))$

And, we continue this procedure, and hopefully we arrive at q. Two issues in this argument:

(i) Can the piecewise geodesics (or broken geodesic) be a single geodesic?

(ii) Can we arrive at q eventually?

To ~~over come~~ ~~figure out th~~ solve this two issues, we argue as below:

~~At the~~ In $B(p,\rho)$, we know the radial geodesic from p to p_0 is $c(t) = \exp_p tV$, for some $V \in T_pM$.

with $p_0 = \exp_p \rho V$. (That is ~~we~~ ~~pa~~ c is parametrized by arc length).

We consider the curve

$$c(t) = \exp_p tV, \quad t \in [0,\infty) \quad \left(\begin{array}{c}\text{by (iv) we can}\\ \text{do this}\end{array}\right)$$

We hope to show ~~$d(c(t))$~~ $c(r) = q$, where $r = d(p,q)$.

If this was shown to be true, we know $c(r)$ is the shortest one and we are done.

In other words, we hope to prove

$$d(c(r), q) = 0. \qquad (1)$$

Note we know

$$d(c(0), q) = d(p, q) = r. \qquad (2)$$

Consider the set

$$I := \{ t \in [0,r], \ d(c(t), q) = r - t \}$$

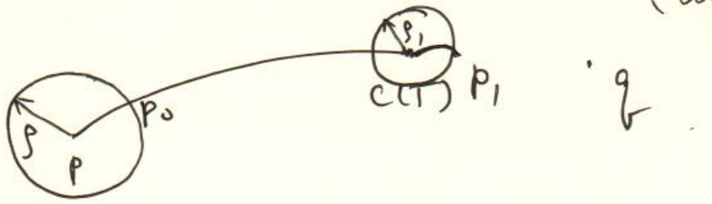
(2) ~~implies~~ $\Leftrightarrow$ $0 \in I$. Hence $I \neq \phi$.

Moreover, since $f(t) := d(c(t), q) - r + t$ is continuous, and $I = f^{-1}(0) \cap [0,r]$, I is closed.

Let $T = \sup_{t \in I} t$. Since I is closed, we see $T \in I$.

If $T = r$. Then we are done.

Suppose $T < r$, consider the normal ball $B(c(T), \rho_1)$ (w.l.o.g., assume $\rho_1 < r - T$.)



Let $p_1 \in \partial B(c(T), \rho_1)$ be the point at which $d(q, \cdot)$ attains its minimum on $\partial B(c(T), \rho_1)$. ~~We will show~~ ϕ

Consider the three points: $c(T)$, p_1, q.

by def, $d(q, c(T)) = r - T$.

We have by triangle ineq,

$$\rho_1 + d(p_1, q) = d(c(T), p_1) + d(p_1, q) \geq \underbrace{d(c(T), q)}_{= r - T} \qquad (3)$$

$$\Rightarrow d(p_1, q) \geq r - T - \rho_1$$

(58)

On the other hand, thinking of any curve γ from $c(T)$ to q. Then exists t s.t. $\gamma(t) \in \partial(B(c(T), \rho_1))$

$$\cancel{\text{Length}(\gamma) \geq d(c(T), p_1) + d(\gamma(t), q)}$$

$$\cancel{\geq d(c(T), \rho_1) + d(p_1, q)}$$

$$\cancel{\geq d + d(p_1, q)}$$

~~This tells~~ $\cancel{d(c(T), q) \geq \rho_1 + d(p_1, q)}$

$$\Rightarrow \text{Length}(\gamma) \geq d(c(T), \gamma(t)) + d(\gamma(t), q)$$

$$= d(c(T), p_1) + d(\gamma(t), q)$$

$$\geq d(c(T), p_1) + d(p_1, q).$$

$$\Rightarrow d(c(T), q) \geq d(c(T), p_1) + d(p_1, q)$$

Combining with (3), we get equality:

$$\boxed{d(c(T), q) = d(c(T), p_1) + d(p_1, q)}$$

$$\Rightarrow \cancel{d(c(T), p_1) = d(c(T), q) - d(p_1, q)}$$

$$\cancel{= r - T -}$$

$$d(p_1, q) = d(c(T), q) - d(c(T), p_1)$$

$$= r - T - \rho_1$$

$$= r - (T + \rho_1).$$

Now if we show $p_1 = \exp_p (T+\rho_1)V = c(T+\rho_1)$, (4)

we have $T + \rho_1 \in I$, which contradicts to the definition of T.

It remains to show (4). We use Proposition 1 to prove it.

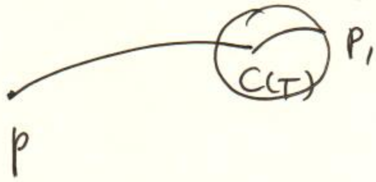
That is, we show the curve

$c|_{[0,T]}$ and the radial geodesic from $c(T)$ to p_1 (5)

is the shortest curve from p to p_1. Note this curve has length $T+\rho_1$.

~~Consider 3 points $p, c(T), p_1$~~

~~$d(p,p_1) \le d(p,c(T)) + d(c(T),p_1)$~~
~~(triangle ineq.)~~



q

Consider the three points p, p_1, q to figure out $d(p,p_1)$.

$$\underbrace{d(p,q)}_{r} \le d(p,p_1) + \underbrace{d(p_1,q)}_{\overset{\shortparallel}{r-T-\rho_1}}$$

$$\Rightarrow \quad T+\rho_1 \le d(p,p_1)$$

Hence the "broken" curve in (5) is the shortest curve from p to q. Then Prop. tells that it is a smooth geodesic when parametrized with arc length.

By uniqueness of geodesics with given initial ~~data~~ values, it has to coincide with c. Therefore

$$p_1 = \exp_p (T+\rho_1)V = c(T+\rho_1).$$ $\boxtimes$

Then we finish the proof of (iv) $\Rightarrow$ (v).

Next, we prove the equivalence of (i) – (iv).

(iv) $\Rightarrow$ (iii) is trivial

(iii) $\Rightarrow$ (ii) Let $K \subset M$ be <u>closed</u> and <u>bounded</u>.

"boundedness" $\Rightarrow K \subset \overline{B(p,r)}$ for some $r>0$.

~~Since K is assumed to be closed.~~

Since $\exp_p$ is defined on all of T_pM, from the proof ~~of~~ for $(iv) \Rightarrow (V)$, we know any $q \in \overline{B(p,r)}$ can be connected to p via ~~exp~~. $c(t) = \exp_p(tV)$, with $c(d(p,q)) = \exp_p(d(p,q)V)$.
for some V

Hence $\overline{B(p,r)}$ is the image of the ~~a~~ image of the compact ball in T_pM of radius r under the continuous map $\exp_p$.

Hence, $\overline{B(p,r)}$ is compact. Since K is assumed to be ~~clo~~ closed, and shown to be contained in a compact set, it must be compact itself. $\square$

$(ii) \Rightarrow (i)$ Let $(p_n)_{n\in\mathbb{N}} \subset M$ be a Cauchy sequence.

~~It then is bounded.~~ If it has an accumulation point x_0, there exists a subsequence $\{x_{i_k}\}$. ~~lim~~ $x_{i_k} \to x_0$, $k \to \infty$.

~~Then~~ $\forall\, \varepsilon > 0$, $\exists\, N$, when $n, l > N$, we have

$$| d(x_n, x_0) - d(x_l, x_0) | \leq d(x_n, x_l) < \varepsilon$$

Let $p_0 \in M$. Since $(p_n)_{n\in\mathbb{N}} \subset M$ is a Cauchy sequence, we have $\forall\, \varepsilon > 0$, $\exists\, N$, when $n, l > N$,

$$| d(p_n, p_0) - d(p_l, p_0) | \leq d(p_n, p_l) < \varepsilon$$

That is $(d(p_n, p_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, then

$$\lim_{n\to\infty} d(p_n, p_0) \text{ exists.} \qquad (*)$$

If $(p_n)_n$ has an accumulation point a_0, i.e. $\exists$ subsequence $(p_{n_{i_k}})$ $p_{i_k} \to a_0$ as $k \to \infty$

Pick $p_0 = a_0$ in $(*)$, we have $\lim_{n\to\infty} d(p_n, p_0) = \lim_{k\to\infty} d(p_{n_k}, p_0) = 0$.

That is, $p_n \to p_0$ as $n \to \infty$.

Otherwise, if (p_n) has no accumulate point, then (p_n) is closed.

Note (p_n) is bounded since it is Cauchy.

By assumption (ii), (p_n) is compact.

But each p_n is not an accumulate point, we have $p_n \in U_n$, $p_i \notin U_n, \forall i \neq n$. Hence $\{U_n\}$ is an open cover of (p_n), with out any finite sub cover. This contradicts to the compactness of (p_n). $\square$

(i) $\Rightarrow$ (iv) ~~Suppose M is not geodesically complete. Then $\exists$ some normal geodesic γ of M is defined for $s < s_0$ and is not defined for s_0. Let $\{s_n\}$ be a convergent sequence, converging to s_0~~

Let γ be a geodesic in M, parametrized by arc length, and being defined on a maximal interval I. I then is not empty. Moreover, by the "local existence and uniqness of geodesics" (ODE theory), we know I is an open interval. Next we show I is closed. Then I has to be $(-\infty, \infty)$.

Let $(t_n)_{n \in \mathbb{N}} \subset I$ be converging to t. Notice that $d(\gamma(t_n), \gamma(t_m)) \leq |t_n - t_m| = \text{arc length } \overline{\gamma(t_n)\gamma(t_m)}$

We know $(\gamma(t_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in M.

$\boxed{\text{M is complete}} \Rightarrow \exists\, p_0 \in M$, $\gamma(t_n) \to p_0$ as $n \to \infty$.

$\exists \delta > 0$, $\exists W \ni p_0$ a totally normal neighborhood of p_0 s.t. $\forall q \in W$, $\exp_q(B(0,\delta)) \supset W$.

There exists N, s.t. when $n, m \geq N$, we have

$$|t_n - t_m| < \delta \qquad (a)$$

$$\text{and} \quad \gamma(t_n), \gamma(t_m) \in W. \qquad (b)$$

By (a), and the property of W, there exist a unique geodesic c from $\gamma(t_n)$ to $\gamma(t_m)$ of length $< \delta$. Therefore c ~~coincides with~~ has to be a subarc of γ. Since $\exp_{\gamma(t_n)}$ is a diffeomorphic on $B(0,\delta) \subset T_{\gamma(t_n)}M$ and $\exp_{\gamma(t_n)}(B(0,\delta)) \supset W$, c extends γ to p_0.

□

Corollary (1) Compact Riemannian manifold is complete.

Pf: closed subset of a compact space is compact. □
Apply. H-R. Thm (ii).

Corollary (2) A closed submanifold of a complete Riemannian manifold is complete in the induced metric. In particular the closed submanifolds of Euclidean space are complete.