§4. Levi-Civita (Riemannian) Connection.

There are too many connections on a given smooth manifold.

~~For~~ Let $X, Y \in \Gamma(TM)$. In a ~~local~~ coordinate neighborhood (U, x).

write $X = X^i(x)\frac{\partial}{\partial x^i}$, $Y = Y^j(x)\frac{\partial}{\partial x^j}$. By definition we have

$$\nabla_X Y = \nabla_{X^i\frac{\partial}{\partial x^i}}\left(Y^j\frac{\partial}{\partial x^j}\right) = X^i\frac{\partial(Y^j)}{\partial x^i}\frac{\partial}{\partial x^j} + X^iY^j\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}.$$

Since $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} \in \Gamma(TU)$, there exist fcts $\left\{\begin{smallmatrix}k\\ ij\end{smallmatrix}\right\}$, $k = 1, 2, \dots, n$ s.t.

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \left\{\begin{matrix}k\\ ij\end{matrix}\right\}\frac{\partial}{\partial x^k}.$$

$$\Rightarrow \nabla_X Y = X(Y^j)\frac{\partial}{\partial x^j} + X^iY^j\left\{\begin{matrix}k\\ ij\end{matrix}\right\}\frac{\partial}{\partial x^k}$$

$$= \left(X(Y^k) + X^iY^j\left\{\begin{matrix}k\\ ij\end{matrix}\right\}\right)\frac{\partial}{\partial x^k}.$$

That is, the connection ∇ is determined by the n^3 smooth fcts.

$$\left\{\begin{matrix}k\\ ij\end{matrix}\right\}.$$

Let $c : [a,b] \to M$ be a curve such that the ~~tangent~~ velocity vector field $\dot{c}(t)$ (along c) is parallel. Then locally, we can write $c(t) = (x^1(t), \dots, x^n(t))$

and
$$0 = \frac{D\dot{c}(t)}{dt} = \frac{d}{dt}\dot{x}^k(t)\left.\frac{\partial}{\partial x^k}\right|_{c(t)} + \dot{x}^j(t)\frac{dx^i}{dt}\left\{\begin{matrix}k\\ ij\end{matrix}\right\}(x(t))\cdot\left.\frac{\partial}{\partial x^k}\right|_{c(t)}$$

$$= \left(\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\left\{\begin{matrix}k\\ ij\end{matrix}\right\}(x(t))\right)\left.\frac{\partial}{\partial x^k}\right|_{c(t)}.$$

$$\Leftrightarrow \quad \ddot{x}^k(t) + \left\{\begin{matrix}k\\ ij\end{matrix}\right\}(c(t))\,\dot{x}^i(t)\,\dot{x}^j(t) = 0, \quad k = 1, \dots, n.$$

Recall the geodesic equations of a Riemannian mfld (M, g) are

Christoffel symbols.

$$\ddot{x}^k(t) + \Gamma^k_{ij(c(t))}\dot{x}^i(t)\dot{x}^j(t) = 0, \quad k=1,\dots,n.$$

We hope to find a connection, under which a geodesic is a curve whose ~~tangent~~ velocity vector field is parallel along it.

That is, we are looking for a connection ∇, s.t.

$$\left\{ {k \atop ij} \right\} = \Gamma^k_{ij} = \frac{1}{2} g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l}).$$

From this aim, we see the connection has to be "compatible" with the Riemannian metric.

Recall that along a geodesic γ, we have $\langle \dot\gamma, \dot\gamma \rangle_g \equiv \text{const}$.

It is natural to require g, as a $(0,2)$-tensor, is parallel ~~under~~ w.r.t. ∇.

i.e.
$$\nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \quad \forall X,Y,Z \in \Gamma(TM)$$

Definition 4. We say ∇ is compatible with g if the Riemannian metric tensor g is parallel. In other words, ∇ is compatible with g if for all $X, Y, Z \in \Gamma(TM)$,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Let us calculate $\left\{ {k \atop ij} \right\}$ of ~~this con~~ such connections.

$$g_{ij,k} = \frac{\partial}{\partial x^k}\left(g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right) = g\left(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right)$$
$$= g\left(\left\{ {l \atop ki} \right\}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \left\{ {l \atop kj} \right\}\frac{\partial}{\partial x^l}\right)$$
$$= g_{lj}\left\{ {l \atop ki} \right\} + g_{il}\left\{ {l \atop kj} \right\}.$$

That is $\quad g_{ij,k} = g_{\ell j}\left\{{\ell \atop ki}\right\} + g_{i\ell}\left\{{\ell \atop kj}\right\}. \qquad (1)$

Permutating indices, we obtain

$$\begin{matrix} i & j & k \\ \downarrow & \downarrow & \downarrow \\ k & i & j \end{matrix} \qquad g_{ki,j} = g_{\ell i}\left\{{\ell \atop jk}\right\} + g_{k\ell}\left\{{\ell \atop ji}\right\} \qquad (2)$$

$$\begin{matrix} i & j & k \\ \downarrow & \downarrow & \downarrow \\ j & k & i \end{matrix} \qquad g_{jk,i} = g_{\ell k}\left\{{\ell \atop ij}\right\} + g_{j\ell}\left\{{\ell \atop ik}\right\} \qquad (3)$$

(1) + (2), (3) give us $\qquad$ using symmetry $g_{ij} = g_{ji}\ \forall i,j$

$$g_{ij,k} + g_{ki,j} - g_{jk,i} \overset{\downarrow}{=} g_{j\ell}\left(\left\{{\ell \atop ki}\right\} - \left\{{\ell \atop ik}\right\}\right) + g_{k\ell}\left(\left\{{\ell \atop ji}\right\} - \left\{{\ell \atop ij}\right\}\right)$$

$$+\, g_{i\ell}\left(\left\{{\ell \atop kj}\right\} + \left\{{\ell \atop jk}\right\}\right)$$

Now <u>if</u> we further have the symmetry

$$\left\{{\ell \atop ki}\right\} = \left\{{\ell \atop ik}\right\}, \quad \forall i, \ell, k. \qquad (\maltese)$$

then

$$g_{ij,k} + g_{ki,j} - g_{jk,i} = 2\, g_{i\ell}\left\{{\ell \atop kj}\right\}$$

$$\Rightarrow \quad \left\{{\ell \atop kj}\right\} \tfrac{1}{2}\, g^{pi}\left(g_{ij,k} + g_{ki,j} - g_{jk,i}\right) = 2\, \underbrace{g^{pi} g_{i\ell}}_{\delta^p_\ell}\left\{{\ell \atop kj}\right\}$$

$$\left\{{p \atop kj}\right\} = \tfrac{1}{2}\, g^{pi}\left(g_{ij,k} + g_{ki,j} - g_{jk,i}\right) = \Gamma^p_{kj}.$$

↳ Christoffel symbol.

Then we obtain the Christoffel symbols!! (That is, under such connections, a geodesic is a curve whose velocity v.f. is parallel

~~More thoughts~~ <u>Express the condition ($\maltese$) in global terms</u>

$$(\maltese) \iff \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = 0. \qquad (*)$$

For $X, Y \in \Gamma(TM)$, $\overset{\text{only}}{,}$ $\nabla_X Y - \nabla_Y X$ is not a tensor.

~~To make~~ The global expression of LHS of $(*)$ is as follows. For $X, Y \in \Gamma(TM)$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

<u>Proposition 8</u> T is a $(1,2)$-tensor.

<u>Proof</u>: T gives the multilinear map

$$T: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$
$$(X,Y) \mapsto T(X,Y)$$

Moreover $T(fX,Y) = T(X,fY) = fT(X,Y)$.

$$\left(\begin{aligned} \text{e.g. } T(fX,Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX,Y] \\ &= f\nabla_X Y - Y(f)X - f\nabla_Y X - fXY + Y(f)X + fYX \\ &= fT(X,Y) - \cancel{Y(f)X} + \cancel{Y(f)}X. \end{aligned} \right)$$

Hence T is a tensor. It is a $(1,2)$-tensor in the sense.

$$T(\omega, X, Y) = \omega(T(X,Y)).$$

<u>Definition 5</u> (Torsion free) We call T the torsion tensor of ∇.
If $T=0$, we call ∇ <u>torsion free</u> (or <u>symmetric</u>) connection.

So, our calculations tell us: A torsion free connection ∇ which is compatible with g has in each coordinate neighborhood.

$$\left\{ {i \atop jk} \right\} = \Gamma^i_{jk}.$$

~~Conversely This means, if such a~~

<u>Definition 6</u> A connection ∇ on (M,g) is called a <u>Levi-Civita connection</u> (also called a <u>Riemannian connection</u>) if it is torsion free, and it is compatible with g.

In this language, our previous calculations tell that if a Levi-Civita connection ~~exist~~ exists on (M,g), it is uniquely determined by the Christoffel symbols.

Conversely, we can define a connection ∇ as follows in each coordinate neighborhood (U, x),

$$\nabla_X Y := \nabla_{X^i \frac{\partial}{\partial x^i}}\left(Y^j \frac{\partial}{\partial x^j}\right) := \left(X^i \frac{\partial Y^k}{\partial x^i} + X^i Y^j \Gamma_{ij}^k\right) \frac{\partial}{\partial x^k}.$$

We can check this is well-defined, and ∇ is ~~a~~ torsion free and is compatible with g. This shows the existence of a Levi-Civita connection on (M, g).

Actually, we prove the following important result.

Theorem 1 (The fundamental theorem of Riemannian geometry)

On any Riemannian manifold (M, g), there exists a unique Levi-Civita connection.

Rmk: This is a remarkable point to note the following observation: On a smooth ~~Riemannian~~ manifold, once we fix a Riemannian metric g, then we get

- a canonical distance function.
- a canonical measure
- a canonical affine connection.

We ~~???~~ in fact already show a proof via local coordinate calculations for Theorem 1. We provide a coordinate-free proof below.

Proof of Thm 1: Assume the Levi-Civita connection ∇ exist. Then we calculate for all $X, Y, Z \in \Gamma(TM)$,

$$g(\nabla_X Y, Z) =: \langle \nabla_X Y, Z \rangle \underset{\nabla g = 0}{\overset{\uparrow}{=}} X(\langle Y, Z\rangle) - \langle Y, \nabla_X Z \rangle$$

$$\overset{\text{torsion free}}{=} X(\langle Y, Z\rangle) - \langle Y, \nabla_Z X + [X,Z] \rangle$$

$$= X(\langle Y, Z\rangle) - \langle Y, \nabla_Z X \rangle - \langle Y, [X, Z] \rangle$$

$$\overset{\nabla g = 0}{=} X(\langle Y, Z\rangle) - \Big(Z\langle Y, X\rangle - \langle \nabla_Z Y, X\rangle \Big) - \langle Y, [X,Z] \rangle$$

$$= X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + \langle \nabla_Z Y, X\rangle - \langle Y, [X,Z]\rangle$$

$$\overset{\text{torsion-free}}{=} X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + \langle \nabla_Y Z, X\rangle + \langle [Z,Y], X\rangle - \langle Y, [X,Z]\rangle$$

$$\overset{\nabla g = 0}{=} X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) - \langle Z, \nabla_Y X\rangle$$

$$+ \langle [Z,Y], X\rangle - \langle Y, [X,Z]\rangle$$

$$\overset{\text{torsion-free}}{=} X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) - \underline{\langle Z, \nabla_X Y\rangle} - \langle Z, [Y,X]\rangle$$

$$+ \langle [Z,Y], X\rangle - \langle Y, [X,Z]\rangle.$$

$\Longrightarrow$

$$\boxed{\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\quad - \langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle + \langle Z, [X,Y]\rangle \qquad (\maltese) \end{aligned}}$$

The RHS is determined by the metric g. So the uniqueness is proved.

For existence, check the $\nabla_X Y$ defined by $(\maltese)$ satisfies all conditions of Levi-Civita connections. $\square$

<u>Remark</u>. The formula $(\maltese)$ is called the <u>Koszul formula</u>

In local coordinate (U, x), let X, Y, Z be $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}$, we will derive the formula for Christoffel symbols Γ_{jk}^i.

Sometimes, using ($\clubsuit$) is more convenient than using Γ_{ij}^k. If in an open subset $U \subset M$, there exists an orthonormal frame field $E_1, \dots, E_n$, (i.e. $\langle E_i, E_j\rangle (p) = \delta_{ij}$, $\forall p \in U$), ($\clubsuit$) gives

$$2\langle \nabla_{E_i} E_j, E_k\rangle = -\langle E_i, [E_j, E_k]\rangle + \langle E_j, [E_k, E_i]\rangle + \langle E_k, [E_i, E_j]\rangle.$$

<u>Exercise</u> 4. Suppose we know the following fact: There exists three vector fields on $S^3 \subseteq \mathbb{R}^4$, i, j, k, which are linearly independent at any point of S^3, such that

$$[i, j] = k, \quad [j, k] = i, \quad [k, i] = j.$$

Assign to S^3 a Riemmannian metric g s.t. i, j, k are orthonormal at any point of S^3.

Calculate the Levi-Civita connection ∇ of g on S^3.

<u>NEXT</u>, we give more geometric interpretations for the two properties of the Levi-Civita connection.

(a) ∇ is compatible with the metric

(b) ∇ is torsion free.

<u>Proposition 9</u> (geometric meaning of (a)). Let M be a smooth mfld with an affine connection ∇. Let g be Rie. metric on M. Then ∇ is compatible with the g <u>iff</u> any <u>parallel transport</u> is an <u>isometry</u>.

<u>Proof</u>: ($\Rightarrow$). Let $c: [a,b] \longrightarrow M$ be a smooth curve with $p = c(a)$. The parallel transport

$$P_{c,a,t}: T_{c(a)}M \to T_{c(t)}M. \quad t \in [a,b]$$

is an isomorphism.

~~Let~~ $\cancel{\{E_i(c(t))\}_{i=1}^{n}}$ ~~be parallel vector~~

Let $V_a, W_a \in T_{c(a)}M$, and $V_t := P_{c,a,t} V_a$
$W_t := P_{c,a,t} W_a$.

Then V_t, W_t are two C^∞ vector fields along c.

· If V_t, W_t can be extended to two C^∞ vector fields on M, we have

$$\underset{\substack{\shortparallel \\ dc(\frac{\partial}{\partial t})}}{\nabla_{\frac{dc}{dt}}} \langle V_t, W_t \rangle \overset{\text{metric compatibility}}{=} \langle \underbrace{\nabla_{\frac{dc}{dt}} V_t}_{\substack{\shortparallel \leftarrow \text{parallel} \\ 0}}, W_t \rangle + \langle V_t, \underbrace{\nabla_{\frac{dc}{dt}} W_t}_{\substack{\shortparallel \leftarrow \text{parallel} \\ 0}} \rangle$$

$$= 0.$$

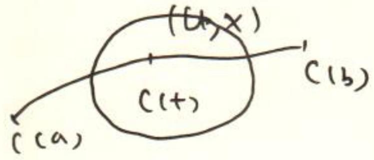
That is $P_{c,a,t}$ preseves the norms ~~and~~ of vectors and angles between vectors.

$\Rightarrow$ $P_{c,a,t}$ is an isometry.

In general, we have to use the following property of <u>induced connection</u>:

$$\boxed{(*1) \quad \frac{d}{dt} \langle V_t, W_t \rangle = \langle \frac{DV_t}{dt}, W_t \rangle + \langle V_t, \frac{DW_t}{dt} \rangle.}$$

<u>Proof</u>: In a ~~local~~ coordinate neighborhood $(U, x^1, \ldots, x^n)$.



$c(t) := (x^1(t), \cdots, x^n(t))$

$$V_t := V^i(t) \frac{\partial}{\partial x^i}\Big|_{c(t)}$$
$$W_t = W^i(t) \frac{\partial}{\partial x^i}\Big|_{c(t)}.$$

We calculate

$$\frac{d}{dt} \langle V_t, W_t \rangle = \frac{d}{dt} \left(V^i(t) W^j(t) \langle \frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)} \rangle \right)$$
$$= \frac{d}{dt} (V^i(t) W^j(t)) \langle \frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)} \rangle$$

$$+ V^i(t)W^j(t)\cdot \frac{d}{dt}\left\langle \frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle$$

$$= \left(\dot V^i(t)W^j(t) + V^i(t)\dot W^j(t)\right)\left\langle \frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle$$

$$+ V^i(t)W^j(t)\underbrace{dc\left(\frac{d}{dt}\right)\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle}_{\parallel \leftarrow \text{compatibility}}$$

$$\left\langle \nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle + \left\langle \frac{\partial}{\partial x^i}, \nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}\right\rangle$$

$$= \left\langle \dot V^i(t)\frac{\partial}{\partial x^i}\Big|_{c(t)} + V^i(t)\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^i}, W^j(t)\frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle$$

$$+ \left\langle V^i(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}, \dot W^j(t)\frac{\partial}{\partial x^j}\Big|_{c(t)} + W^j(t)\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}\right\rangle$$

$$= \left\langle \frac{DV_t}{dt}, W_t\right\rangle + \left\langle V_t, \frac{DW_t}{dt}\right\rangle.$$

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$(\Leftarrow)$ see (91) additional page.

Rmk (*1) In fact (*1) is a general propert of the induced connection $\tilde\nabla$ of a ~~Levi-Civita connection~~ ∇ compatible with g.
Let $\varphi: N \to M$. C^∞. $u \in T_xN$, Let V, W is two smooth vector fields along φ, then

$$\langle \tilde\nabla_u V, W\rangle + \langle V, \tilde\nabla_u W\rangle = u\langle V, W\rangle.$$

Proposition 10: (geometric meaning of (b))

Let ∇ be a torsion-free connection of M.

Let $s: \underset{(x,y)}{\mathbb{R}^2} \to M$ be a C^∞ map. (a "parametrized surface" in M)

Let $V(x,y) \in T_{s(x,y)}M$ be a vector field along s.

For convenience, let us denote

$$ds\left(\frac{\partial}{\partial x}\right) := \frac{\partial s}{\partial x}, \quad ds\left(\frac{\partial}{\partial y}\right) = \frac{\partial s}{\partial y}.$$

Then for the induced connection $\tilde\nabla$,

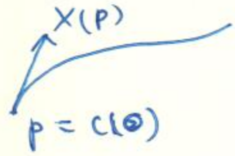
$$\tilde\nabla_{\frac{\partial}{\partial x}} V(x,y) = \left(\frac{DV}{\partial x}\right)_{(x,y)}$$

($\Leftarrow$) i.e. any parallel transport is an isometry

$\Rightarrow$ ∇ is compatible with g.

For any $X, Y, Z \in \Gamma(TM)$. Look at $X(p)$

$$X(\langle Y, Z\rangle) = X(p)\langle Y, Z\rangle$$



Let $c : [0,1] \to M$

$c(0) = p, \quad \dot{c}(0) = X(p)$

We have $X\langle Y, Z\rangle = \dfrac{d}{dt}\Big|_{t=0} \langle Y(c(t)), Z(c(t))\rangle$

let $\{E_1, \dots, E_n\}$ is an orthonormal basis of T_pM.

and $\{E_1(t), \dots, E_n(t)\}$ is given by $E_i(t) = P_{c,t} E_i$.

Since $P_{c,t}$ is isometry. $\{E_i(t)\}$ is orthornormal $\forall t$.

$$\Rightarrow \langle Y(c(t)), Z(c(t))\rangle = \langle Y^i(t) E_i(t), Z^j(t) E_j(t)\rangle$$

$$= Y^i(t) Z^j(t)\, \delta_{ij} = \sum_i Y^i(t) Z^i(t).$$

$$\Rightarrow X\langle Y, Z\rangle = \sum_i \frac{d}{dt}\Big|_{t=0} \left(Y^i(t) Z^i(t)\right) = \sum_i \frac{dY^i}{dt}(0) Z^i(0) + \sum_i Y^i(0) \frac{dZ^i(0)}{dt}$$

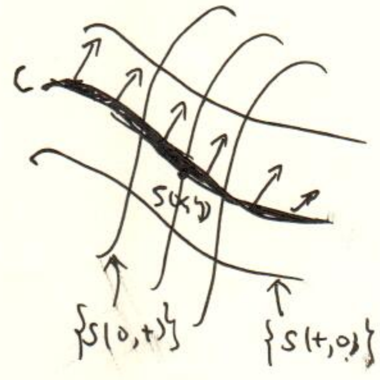
$$= \left\langle \frac{DY}{dt}\Big|_{t=0}, Z\right\rangle + \left\langle Y, \frac{DZ}{dt}\Big|_{t=0}\right\rangle$$

$$= \langle \nabla_{X(p)} Y, Z\rangle + \langle Y, \nabla_{X(p)} Z\rangle.$$

This show ∇ is compatible with g $\square$

can be considered as the <u>covariant derivative</u> along $c(t) := s(t,y)$ of the vector field $t \mapsto V(t,y)$ along c, evalued at $t=x$.

Similarly, we have $\tilde{\nabla}_{\frac{\partial}{\partial y}} V = \frac{DV}{\partial y}$.



Then, we have

$$\boxed{(*2) \quad \frac{D}{\partial x}\frac{\partial s}{\partial y} = \frac{D}{\partial y}\frac{\partial s}{\partial x}.}$$

<u>Remark</u>: In symbols of induced connection, (*2) can be written as $\tilde{\nabla}_{\frac{\partial}{\partial x}} ds\left(\frac{\partial}{\partial y}\right) = \tilde{\nabla}_{\frac{\partial}{\partial y}} ds\left(\frac{\partial}{\partial x}\right)$.

In case $ds\left(\frac{\partial}{\partial x}\right)$, $ds\left(\frac{\partial}{\partial y}\right)$ are both vector fields on M, (e.g., when s is an embedding), (*2) is equivalent to say

$$\nabla_{ds\left(\frac{\partial}{\partial x}\right)} ds\left(\frac{\partial}{\partial y}\right) = \nabla_{ds\left(\frac{\partial}{\partial y}\right)} ds\left(\frac{\partial}{\partial x}\right).$$

This is equivalent to the torsion free property since

$$\left[ds\left(\frac{\partial}{\partial x}\right), ds\left(\frac{\partial}{\partial y}\right)\right] = ds\Big(\underbrace{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]}_{0}\Big) = 0.$$

<u>Proof</u>: Express both sides in a coordinate neighborhood as.

$(U, x^1, \cdots, x^n)$, $\quad s = (s^1, \cdots, s^n)$

$$\frac{\partial s}{\partial y} = ds\left(\frac{\partial}{\partial y}\right) = \frac{\partial s^i}{\partial y}\frac{\partial}{\partial s^i}$$

$$\frac{\partial s}{\partial x} = ds\left(\frac{\partial}{\partial x}\right) = \frac{\partial s^i}{\partial x}\frac{\partial}{\partial s^i}$$

$$\Rightarrow \frac{D}{\partial x}\frac{\partial s}{\partial y} = \frac{\partial^2 s^i}{\partial x \partial y}\frac{\partial}{\partial s^i} + \frac{\partial s^i}{\partial y}\nabla_{ds\left(\frac{\partial}{\partial x}\right)}\frac{\partial}{\partial s^i}$$

$$= \frac{\partial^2 s^i}{\partial x \partial y} \cdot \frac{\partial}{\partial s^i} + \frac{\partial s^i}{\partial y} \frac{\partial s^j}{\partial x} \nabla_{\frac{\partial}{\partial s^j}} \frac{\partial}{\partial s^i}$$

Similarly,

$$\frac{D}{\partial y} \frac{\partial s}{\partial x} = \frac{\partial^2 s^i}{\partial y \partial x} \frac{\partial}{\partial s^i} + \frac{\partial s^i}{\partial y} \frac{\partial s^j}{\partial x} \nabla_{\frac{\partial}{\partial s^i}} \frac{\partial}{\partial s^j}.$$

Then the proposition follows from the fact that

$$\frac{\partial^2 s^i}{\partial y \partial x} = \frac{\partial^2 s^i}{\partial x \partial y} \quad \& \quad \underbrace{\nabla_{\frac{\partial}{\partial s^j}} \frac{\partial}{\partial s^i} - \nabla_{\frac{\partial}{\partial s^i}} \frac{\partial}{\partial s^j}}_{\text{torsion free}} = \left[\frac{\partial}{\partial s^j}, \frac{\partial}{\partial s^i}\right] = 0$$

$\square$

<u>Remark (*2)</u>: (*2) is also a general property of an induced connection $\tilde{\nabla}$ of a torsion free connection ∇.

Let $\varphi : N \to M$ C^∞, X, Y be two C^∞ vector fields on N. ~~along~~ Then $d\varphi(X), d\varphi(Y)$ are C^∞ vector fields along φ, then $\tilde{\nabla}_X d\varphi(Y) - \tilde{\nabla}_Y d\varphi(X) = d\varphi([X, Y])$.

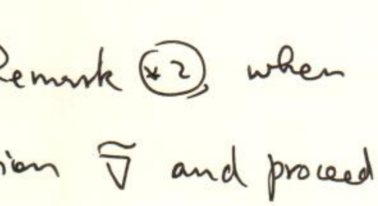
By Remark (*1) (p. 91) and the above Remark (*2), when doing calculations, we can suppress the notation $\tilde{\nabla}$ and proceed formally as if vector fields along φ were actually defined on M.

<u>Exercise 5</u>: (Variation of the energy functional: A coordinate-free calculation).

Let $\gamma : [a,b] \to M$ be a C^∞ curve, and

$\alpha : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a variation.

Recall



$$E(\gamma) := \frac{1}{2} \int_a^b \langle \dot\gamma(t), \dot\gamma(t) \rangle \, dt$$

$$= \frac{1}{2} \int_a^b \langle d\gamma(\tfrac{d}{dt}), d\gamma(\tfrac{d}{dt}) \rangle \, dt.$$

Show that

$$\left.\frac{dE(\alpha(s))}{ds}\right|_{s=0} = -\int_a^b \left\langle \underset{\frac{\partial\alpha}{\partial s}}{\overset{\shortparallel}{d\alpha\left(\frac{\partial}{\partial s}\right)(0,t)}},\ \frac{D\dot\gamma(t)}{dt} \right\rangle dt$$

$$- \left\langle \frac{\partial\alpha}{\partial s}(0,a),\ \frac{D\dot\gamma}{dt}(a) \right\rangle$$

$$+ \left\langle \frac{\partial\alpha}{\partial s}(0,b),\ \frac{D\dot\gamma}{dt}(b) \right\rangle .$$

Hint: Using Proposition 10.

Recall a calculation in coordinates has be carried out in our discussions in Chapter II. Geodesics.

Final Remark. Recall that any Rie. mfld M can be embedded to the standard Euclidean space E isometrically. In the Euclidean space the Levi-Civita connection $\bar\nabla$ is given by the directional derivatives. So for any $X, Y \in \Gamma(TM)$, X, Y can also be ~~considered as~~ extended to vector fields $\bar X, \bar Y$ on E. (at least locally around M.). ~~Then~~ But usually

$$\bar\nabla_{\bar X}\bar Y(p) \in T_pE$$

is not lie in ~~the~~ T_pM any more. ~~We can~~ The orthogonal projection $\pi: T_pE \to T_p\cancel{E}M$ gives

$$\pi \circ (\bar\nabla_{\bar X}\bar Y) \in T_pM$$

One can check that $\pi(\bar\nabla_{\bar X}\bar Y)$ gives a Levi-Civita connection on M w.r.t. the induced metric from E.