(IV) Curvature

The Riemannian curvature tensor was introduced by Riemann in his 1854 lecture as a natural invariant for what is called the equivalence problem in Riemannian geometry. This problem, comes out of the problem one faces when writing the same metric in two different coordinates. Namely, how is one to know that they are the same or equivalent. The idea is to find invariants of the metric that can be computed in coordinates and then try to show that two metrics are equivalent if their invariant expressions are equal. This problem was further clarified by Christoffel.

Our previous discussions on geodesics and connections follow roughly the historical development. However, we will have a discussion on curvature tensor different from it's historical development. (Notice that the idea of a connection postdates Riemann's introduction of the curvature tensor).

On retrospection of our previous discussions, roughly speaking, " the first variation (of length or energy) gives the connection ". In this chapter, we will see, roughly speaking, " the second variation gives the curvature " !

§1. The Second Variation.

We already know that a geodesic is not necessarily ~~a shor~~ minimizing. Is a geodesic a "local minima"? One way to explore this property is to calculate the second variation of length or energy. (Recall from II.5 and Exercise 6.2, among curves $\in C_{p,q}$ (piecewise smooth curves from p to q), a geodesic is characterized as the critical point of the energy functional).

~~Let~~ Let $\gamma : [a,b] \to M$ be a normal geodesic, i.e. $|\dot{\gamma}(t)| \equiv 1$. We consider a 2-parameter variation F of γ. That is, a smooth map

$$F : [a,b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to M.$$

such that $F(t,0,0) = \gamma(t)$.

Let $E(v,w)$ be the energy of the curve $\gamma_{v,w}(t) := F(t,v,w)$.

And
$$V(t) = \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$
are the two corresponding variational fields.

Recall.
$$\frac{\partial}{\partial w} E(v,w) = \frac{1}{2}\int_a^b \frac{\partial}{\partial w} \left\langle \frac{\partial F}{\partial t}(t,v,w), \frac{\partial F}{\partial t}(t,v,w) \right\rangle dt$$

$$\overset{\text{compatibility}}{=} \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle dt$$

$$\overset{\text{torsion-free}}{=} \int_a^b \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle dt$$

Then
$$\frac{\partial^2}{\partial w \partial v} E(v,w) = \int_a^b \frac{\partial}{\partial w} \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle dt$$

$$\overset{\text{compatibility}}{=} \int_a^b \left(\left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \tilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial t} \right\rangle \right) dt$$

$$\overset{\text{torsion-free}}{=} \int_a^b \left(\left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \tilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle \right) dt$$

Restricting the above equality to the curve γ, i.e. to the case $v = w = 0$:

$$\frac{\partial^2}{\partial w \partial v}\bigg|_{v=w=0} E(v,w)$$

$$= \int_a^b \left(\left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \dot{\gamma}(t) \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} W(t) \right\rangle \right) dt$$

Now, we hope to make use of the fact that γ is a geodesic, i.e. $\tilde{\nabla}_{\frac{\partial}{\partial t}} \dot{\gamma}(t) = 0$. For that purpose, we hope to interchange the order of the covariant derivative $\tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}}$. Hence we proceed:

$$\frac{\partial^2}{\partial w \partial v}\bigg|_{(0,0)} E(v,w) = \int_a^b \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle dt$$

$$+ \int_a^b \left(\left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} W(t) \right\rangle \right) dt$$

$$=: \mathrm{I} + \mathrm{II}.$$

Then the second term becomes

$$\mathrm{II} \overset{\text{compatibility}}{=} \int_a^b \left(\frac{\partial}{\partial t} \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle - \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \underbrace{\tilde{\nabla}_{\frac{\partial}{\partial t}} \dot{\gamma}(t)}_{=0} \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V, \tilde{\nabla}_{\frac{\partial}{\partial t}} W \right\rangle \right) dt$$

$$= \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle \bigg|_a^b + \int_a^b \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} W(t) \right\rangle dt.$$

Therefore, we obtain the following <u>Second Variation Formula</u>:

$$\boxed{\begin{aligned}&\frac{\partial^2}{\partial w \partial v}\Big|_{(0,0)} E(v,w) = \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle\Big|_a^b \\ &(SVF) \qquad + \int_a^b \left(\left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} W(t) \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle \right) dt\end{aligned}}$$

Rmk 1: Usually, we will suppress the notation $\tilde{\nabla}$ and proceed formally, as if vectors along γ were actually defined on M and write

$$\frac{\partial^2}{\partial w \partial v}\Big|_{(0,0)} E(v,w) = \langle \nabla_W V, \dot{\gamma} \rangle\Big|_a^b + \int_a^b \left(\langle \nabla_{\dot{\gamma}} V, \nabla_{\dot{\gamma}} W \rangle + \langle \nabla_W \nabla_{\dot{\gamma}} V - \nabla_{\dot{\gamma}} \nabla_W V, \dot{\gamma} \rangle \right) dt$$

□

Rmk 2: In particular, (SVF) tells

$$\frac{d^2}{dv^2}\Big|_{v=0} E(v) =: E''(0) = \left\langle \tilde{\nabla}_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \right\rangle\Big|_a^b$$

$$+ \int_a^b \left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial v}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \right\rangle dt$$

For proper variations, (i.e. $V(a) = V(b) = 0$), or, generally, when $\tilde{\nabla}_{\frac{\partial}{\partial v}} V(t) = 0$ at $t = a$ and $t = b$, we have

$$E''(0) = \int_a^b \left(\underbrace{\left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) \right\rangle}_{\geq 0} + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial v}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \right\rangle \right) dt$$

Now we see the sign of the 2^{nd} term is very important to decide the sign of $E''(0)$, which is useful to ~~dif~~ decide whether a geodesic has a locally minimal ~~energy~~ energy functional Θ (for curves parametrized proportionally to arc length, equivalent to a locally minimal arc length).

In particular, if the 2^{nd} term vanish, (or ≥ 0), ~~then~~ we have $E''(0) \geq 0$ and the local minimum is guaranteed.

~~Since~~ In $\mathbb{R}^n$ (a flat case), any geodesic is minimizing.

From that sense, the term

$$\left\langle \widetilde{\nabla}_{\frac{\partial}{\partial v}} \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial v}} V(t), \dot{\gamma}(t) \right\rangle$$

play ~~a~~ the role of "curvature". $\square$

Consider variations with the property $\left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle \Big|_a^b = 0$,

(e.g., proper variations), we have

$$\boxed{\frac{\partial^2}{\partial w \partial v}\Big|_{(0,0)} E(v,w) = \int_a^b \left(\left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \widetilde{\nabla}_{\frac{\partial}{\partial t}} W \right\rangle + \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial w}} \widetilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot{\gamma}(t) \right\rangle \right) dt.}$$

$$:= I(V(t), W(t)).$$

We will see this quantity ~~will~~ plays a central rôle in our subsequent discussions about curvature-related geometries.

The second term in $I(V,W)$ suggests to define for $X, Y, Z \in \Gamma(TM)$,

$$\bar{R}(X,Y)Z :\overset{?}{=} \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \quad \in \Gamma(TM).$$

But $\bar{R}(X, fY)Z = \nabla_X(f\nabla_Y Z) - f\nabla_Y \nabla_X Z$

$$= X(f)\nabla_Y Z + f\left[\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z\right]$$

i.e. $\bar{R}$ is not a tensor !!

~~Definition. (Riemannian Curvature tensor) We d~~

~~Let ∇ be an affine connection~~

We define for $X, Y, Z \in \Gamma(TM)$

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It gives a multilinear map

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM).$$

Proposition 1. R is a $(1,3)$-tensor.

Proof: Notice that this time

$$R(X,fY)Z = \nabla_X(f(\nabla_Y Z)) - f\nabla_Y\nabla_X Z - \nabla_{[X,fY]}Z$$
$$= X(f)\nabla_Y Z + f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z)$$
$$- \nabla_{X(f)Y + f[X,Y]}Z$$
$$= X(f)\nabla_Y Z + f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z)$$
$$- X(f)\nabla_Y Z - f\nabla_{[X,Y]}Z$$
$$= fR(X,Y)Z.$$

One can further check that

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = fR(X,Y)Z.$$

Hence R is a tensor. We say it is a $(1,3)$-tensor, We actually mean $R(\omega, X, Y, Z) = \omega(R(X,Y)Z)$. $\square$

We will call R the ~~Riemannian~~ curvature tensor.

Rmk (1) The curvature tensor is well-defined for any affine connection on M.

(2) Notice that X, Y appear skew-symmetrically in $R(X,Y)Z$ (i.e. $R(X,Y)Z = -R(Y,X)Z$.) while Z plays its own role on top of the line, hence we use the usual notation $R(X,Y)Z$, instead of $R(X,Y,Z)$.

(3) Some textbooks (e.g. [doC]) adopt a different sign in the definition of R, (i.e. $\tilde{R}(X,Y)Z) = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$). One should always first check the authors' notation for curvature tensor when reading works on Riemannian geometry unfortunately.

(4) (Locality): At $p \in M$, $R(X,Y)Z(p)$ only depends on $X(p), Y(p), Z(p) \in T_pM$. This is due to the tensorial property

$$R(X,Y)Z(p) = X^i Y^j Z^k R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}.$$

Now Let's come back to the SVF:

$$I(V,W) = \int_a^b \left(\left\langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} W(t) \right\rangle + \left\langle \tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t), \dot\gamma(t) \right\rangle \right) dt.$$

~~Proposition 2: We have~~

$$\cancel{\tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t) = R(W(t), \dot\gamma(t)) V(t).}$$

Proposition 2: Let $s: \mathbb{R}^2 \to M$ be a parametrized surface, and V a C^∞ vector field along s. Then

$$\frac{D}{\partial x}\frac{D}{\partial y} V - \frac{D}{\partial y}\frac{D}{\partial x} V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V. \quad (*)$$

(In another notations)

$$\left(\text{or}, \ \tilde{\nabla}_{\frac{\partial}{\partial x}} \tilde{\nabla}_{\frac{\partial}{\partial y}} V - \tilde{\nabla}_{\frac{\partial}{\partial y}} \tilde{\nabla}_{\frac{\partial}{\partial x}} V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V \right).$$

Sketch of proof: First by the Locality remark above, at each point $s(x,y) \in M$, the RHS is well defined since

$$\frac{\partial s}{\partial x} = ds\left(\frac{\partial}{\partial x}\right), \ \frac{\partial s}{\partial y}, V \in T_{s(x,y)}M.$$

Then $(*)$ can be proved by computing in a coordinate neighborhood

Def of Both sides are clear: pick $(U, x^1, \dots, x^n)$

$$s(x,y) = (s^1(x,y), \dots, s^n(x,y))$$

$$R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V = \frac{\partial s^i}{\partial x}\frac{\partial s^j}{\partial y} V^k \cdot R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}$$

while in LHS,

$$\frac{D}{\partial x}\frac{D}{\partial y} V = \frac{D}{\partial x}\left(\frac{\partial V^i(x,y)}{\partial y}\frac{\partial}{\partial x^i} + V^i \nabla_{ds(\frac{\partial}{\partial y})} \frac{\partial}{\partial x^i} \right)$$

$$= \frac{D}{\partial x}\left(\frac{\partial v^i}{\partial y}\frac{\partial}{\partial x^i} + v^i \frac{\partial s^j}{\partial y} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \right)$$

$$= \frac{\partial^2 v^i}{\partial x \partial y}\frac{\partial}{\partial x^i} + \frac{\partial v^i}{\partial y} \cdot \frac{\partial s^j}{\partial x} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x}\left(v^i \frac{\partial s^j}{\partial y}\right) \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}$$

$$+ v^i \frac{\partial s^j}{\partial y}\frac{\partial s^l}{\partial x} \nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} .$$

So the meaning of each term is clear. The equality (*) then just follows from direct computation. □

In particular, Proposition 2 tells.

$$\tilde{\nabla}_{\frac{\partial}{\partial w}} \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t) - \tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\nabla}_{\frac{\partial}{\partial w}} V(t) = R(W(t), \dot\gamma(t)) V(t) .$$

and hence

$$I(V,W) = \int_a^b \langle \tilde{\nabla}_{\frac{\partial}{\partial t}} V(t), \tilde{\nabla}_{\frac{\partial}{\partial t}} W(t) \rangle + \langle R(W(t), \dot\gamma(t)) V(t), \dot\gamma(t)\rangle \, dt$$

$$\left(= \int_a^b \langle \nabla_{\dot\gamma(t)} V(t), \nabla_{\dot\gamma(t)} W(t) \rangle + \cancel{\langle R(W(t), \dot\gamma(t))} \langle R(W,\dot\gamma)V, \dot\gamma\rangle(t) \, dt . \right)$$

If we further denote $T = \dot\gamma$, then we have

$$\boxed{I(V,W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle + \langle R(W,T)V, T\rangle \right) dt}$$

In particular, $I(V,V) = \int_a^b \left(\langle \nabla_T V, \nabla_T V\rangle + \langle R(V,T)V, T\rangle \right) dt$.

§2. Properties of Curvature tensor : Geometric meaning and Symmetries.

Curvature tensor measures the non-commutativity of the covariant ~~derives~~ derivatives.

§2.1 $\boxed{\text{Ricci Identity}}$: Recall for $f \in C^\infty(M)$, it's ~~hea~~ Hessian $\nabla^2 f$ is symmetric (for torsion free connection), i.e.

$$\nabla^2 f(X,Y) = \nabla^2 f(Y,X) .$$

~~Let ∇~~ For any tensor field $\phi \in \Gamma(\otimes^{r,s}TM)$, we can define

$$R(X,Y)\phi = \nabla_X\nabla_Y\phi - \nabla_Y\nabla_X\phi - \nabla_{[X,Y]}\phi.$$

It is obvious that

$$R(X,Y)f = X(Yf) - Y(Xf) - [X,Y]f = 0.$$

So we can write (for ~~symmet~~ torsion-free connection ∇)

$$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = R(Y,X)f = -R(X,Y)f.$$

We can further check the case $\phi = Z \in \Gamma(TM)$,

$$\nabla^2 Z(X,Y) = \nabla_Y(\nabla Z)(X) = \nabla_Y(\nabla_X Z) - \nabla Z(\nabla_Y X)$$

$$= \nabla_Y\nabla_X Z - \nabla_{\nabla_Y X} Z.$$

Hence
$$\nabla^2 Z(X,Y) - \nabla^2 Z(Y,X) = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{\nabla_Y X} Z + \nabla_{\nabla_X Y} Z$$

$$= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]} Z$$

$$= R(Y,X)Z = -R(X,Y)Z.$$

~~The~~ It is direct to check the general case.

<u>Proposition 3</u> (Ricci Identity). $\forall X, Y \in \Gamma(TM)$, $\phi \in \Gamma(\otimes^{r,s}TM)$, we have

$$\nabla^2\phi(\cdots, X, Y) - \nabla^2\phi(\cdots, Y, X) = R(Y,X)\phi(\cdots) = -R(X,Y)\phi(\cdots).$$

<u>Rmk</u>. ~~①~~ In Euclidean space $\mathbb{R}^n$, pick the directional derivative as the covariant derivative. One easily check that $R(X,Y)$ vanishes. In $\mathbb{R}^n$, we can interchange the orders of taking derivatives freely. However, this is ~~usually~~ not true any more when R is nontrivial. $\square$