§2.2. $\boxed{\text{Geometric meaning}}$: A test case [Spivak II. Chap. 6. Thm 10]

The Ricci identity from last subsection provides an explanation of the curvature tensor from a viewpoint of analysis: it is a term measuring the non-commutativity of taking covariant derivatives.

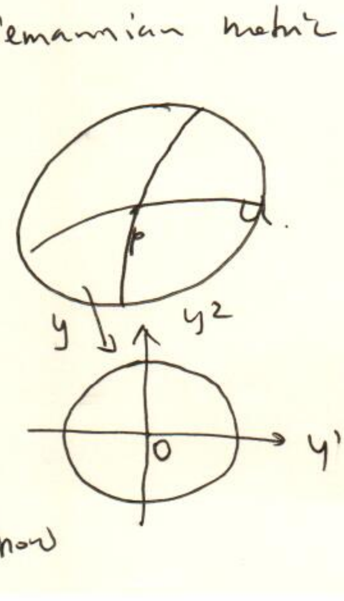
We persue for a geometric meaning of the curvature tensor. Back to Riemann's "equivalence problem": If we know a Riemannian metric $g = g_{ij}\, dx^i \otimes dx^j$ gives $R = 0$, is there a coordinate change $x \mapsto y$ s.t. $g = \sum_i dy^i \otimes dy^i$?

(Or, does $R=0$ implies locally isometric to $(\mathbb{R}^n, \langle,\rangle)$?)

The answer is Yes!

Theorem 1 [Spivak II, Chap. 6, Thm 10] Let (M, g) be an n-dim Riemannian manifold for which the curvature tensor R (for the Levi-Civita connection) is 0. Then M is locally isometric to $\mathbb{R}^n$ with its ~~stand~~ canonical Riemannian metric.

Proof: Let $p \in M$. Pick a coordinate neighborhood $(U, y^1, \dots, y^n)$



~~We claim that we can find vector fields~~

Let $g = g_{ij}\, dy^i \otimes dy^j$.

To prove Thm 1, it is ~~enough to~~ equivalent to show there exist open set $V \subseteq U$, and a coordinate change $x : V \to \mathbb{R}^n$ s.t

$$g = \sum_i dx^i \otimes dx^i .$$

So, w.l.o.g., we can assume we are in $\mathbb{R}^n$, with $y^1, \cdots, y^n$ the standard coordinate system, with a metric

$$g = g_{ij}\, dy^i \otimes dy^j$$

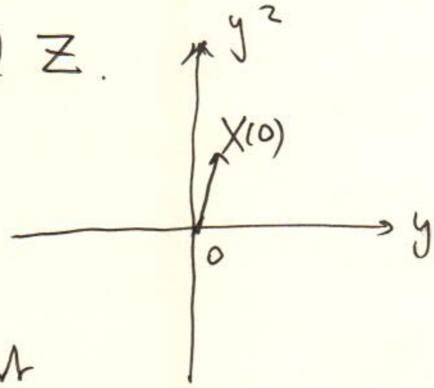
and ∇ be the corresponding Levi-Civita connection.

Step 1. We claim that we can find vector fields X, with arbitrary initial values $X(0) \in T_0 \mathbb{R}^n$. satisfying

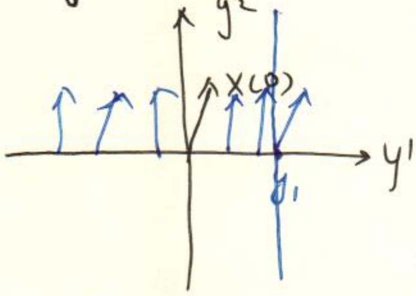
$$\nabla_{\frac{\partial}{\partial y^i}} X = 0 \quad \text{for all } i,$$

and hence

$$\nabla_Z X = 0 \quad \text{for all } Z.$$



To do this, we first choose the curve $y \mapsto (y, 0, \cdots, 0)$, and choose $X(y, 0, \cdots, 0)$ to be the parallel transport along $y \mapsto (y, 0, \cdots, 0)$. (i.e., along the y^1-axis)



Then for each fixed y_1, we choose the curve

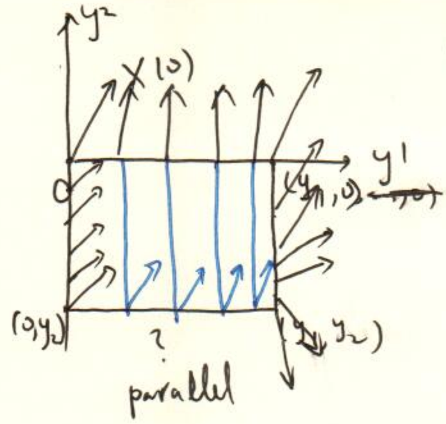
$y \mapsto (y_1, y, 0, \cdots, 0)$

With $X(y^1, 0, \cdots 0)$ as the initial value,

We obtain $X(y^1, y, 0 \cdots, 0)$ via parallel transport along $y \mapsto (y_1, y, 0, \cdots, 0)$

Now we have a vector field X defined on the surface

$$S(y^1,y^2) = (y^1,y^2,0,\cdots,0)$$



By construction, we have

$$\tilde{\nabla}_{\frac{\partial}{\partial y^2}} X = 0 \quad \text{along } S$$

while
$$\tilde{\nabla}_{\frac{\partial}{\partial y^1}} X = 0 \quad \text{along } \{S(y,0)\}$$

Question: Does $\tilde{\nabla}_{\frac{\partial}{\partial y^1}} X$ vanish along S ?

Now we use

$$\tilde{\nabla}_{\frac{\partial}{\partial y^1}} \underbrace{\tilde{\nabla}_{\frac{\partial}{\partial y^2}} X}_{=0} - \tilde{\nabla}_{\frac{\partial}{\partial y^2}} \tilde{\nabla}_{\frac{\partial}{\partial y^1}} X = R\left(\frac{\partial S}{\partial y^1}, \frac{\partial S}{\partial y^2}\right) X = 0$$

$$\Leftrightarrow \quad \tilde{\nabla}_{\frac{\partial}{\partial y^2}}\left(\tilde{\nabla}_{\frac{\partial}{\partial y^1}} X\right) = 0 \qquad (*)$$

Since $\tilde{\nabla}_{\frac{\partial}{\partial y^1}} X \Big|_{y_2=0} = 0$; i.e. $\tilde{\nabla}_{\frac{\partial}{\partial y^1}} X$ is parallel along $\{S(y,0)\}$,

We have by $(*)$ $\tilde{\nabla}_{\frac{\partial}{\partial y^1}} X = 0$ along S.

We can continue in this way to obtain the desired X. This proves the claim.

Now at 0, we can choose $X_1^{(0)},\cdots,X_n^{(0)}$ as orthonormal w.r.t. the metric g. ~~By property of parallel transport, they~~ And construct $X_1,\cdots,X_n$ in the above way. By property of parallel transport, they are orthonormal everywhere.

Step 2: Since ∇ is torsion free, we have

$$0 = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j]$$

By construction, $\nabla_{X_i} X_j = \nabla_{X_j} X_i = 0$. Therefore, we obtain

$$[X_i, X_j] = 0, \quad \forall\, i,j.$$

This means that there is a coordinate system $x^1, \dots, x^n$ with $X_i = \frac{\partial}{\partial x^i}$. (Frobenius theorem in "differential manifold" course. $[X_i, X_j] = 0$ implies integrability.)

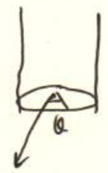
Step 3. Since $\cancel{X_1, \dots, X}$ $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are orthonormal everywhere, we have

$$g = \sum_i dx^i \otimes dx^i. \qquad \square$$

Rmk. (1) In some sense, the flatness $R=0$ is a kind of integrability condition.

It is not true that $R=0$ implies M is globalley isometric to $\mathbb{R}^n$.

Example: $S^1 \times \mathbb{R}^1$ cylinder is the product of a unit circle S^1 and $\mathbb{R}^1$.



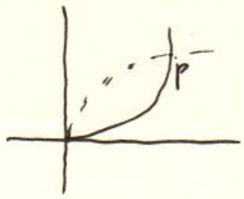
~~In the coordinate (cos.~~

$\forall\, p = (x, y, z) \in S^1 \times \mathbb{R}^1$, we can write $x = \cos\theta$, $y = \sin\theta$, $z = z$, $0 \le \theta < 2\pi$.

Therefore in the coordinate neighborhood $\{(\theta, z) \mid 0 < \theta < 2\pi,\ z \in \mathbb{R}^1\}$, we have the induced metric $g = d\theta \otimes d\theta + dz \otimes dz$.

Hence cylinder has $R = 0$.

Corollary 1. If we can find n everywhere linearly independent vector fields $X_1, \dots, X_n$, which are parallel (i.e. $\nabla_Z X_i = 0, \forall Z$) then the manifold is flat.



Parallel translation of a vector along a closed curve generally bring it back to a different vector.

§2.3 Biandi Identities

Before continuing the discussions of the geometric aspect of the curvature tensor, we prepare symmetry properties of the curvature tensor in this section. We will work on a smooth manifold with a symmetric (i.e. torsion-free) connection ∇.

Proposition 4: The curvature tensor satisfies the following identities: For any $X, Y, Z, W \in \Gamma(TM)$,

(1) $R(X,Y)Z = -R(Y,X)Z$.

(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(The First Bianchi identity)

(3) $(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0$

(The Second Bianchi identity).

Rmk: Let T be any mapping with 3 vector field variables and values that can be added. Summing over cyclic permutations of the variables gives us a new map.

$$\mathfrak{S}\, T(X,Y,Z) = T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y)$$

For example, the Jacobi identity for vector fields can be written as

$$\mathfrak{S}\, [X,[Y,Z]] = 0.$$

In this way, the First Bianchi identity is $\mathfrak{S}\, R(X,Y)Z = 0$ while the second Bianchi identity is

$$\mathfrak{S}\, (\nabla_X R)(Y,Z)W = 0. \qquad \square$$

<u>Proof</u>: (1) is obvious from the definition

(2). $\mathfrak{S}\, R(X,Y)Z = \mathfrak{S}\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \right)$

$$= \mathfrak{S}(\nabla_X \nabla_Y Z) - \mathfrak{S}(\nabla_Y \nabla_X Z) - \mathfrak{S}(\nabla_{[X,Y]} Z)$$

$$= \mathfrak{S}(\nabla_Z \nabla_X Y) - \mathfrak{S}(\nabla_Z \nabla_Y X) - \mathfrak{S}(\nabla_{[X,Y]} Z)$$

$$= \mathfrak{S}\left(\nabla_Z(\nabla_X Y - \nabla_Y X)\right) - \mathfrak{S}(\nabla_{[X,Y]} Z)$$

$$\overset{\text{torsion free}}{=} \mathfrak{S}(\nabla_Z [X,Y]) - \mathfrak{S}(\nabla_{[X,Y]} Z)$$

$$\overset{\text{torsion free}}{=} \mathfrak{S}([Z,[X,Y]])$$

$$\overset{\text{Jacobi identity}}{=} 0$$

(3). Denote

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

then $(\nabla_Z R)(X,Y)W$

$$= \underline{\nabla_Z (R(X,Y)W)} - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W \underline{- R(X,Y)(\nabla_Z W)}$$

$$= [\nabla_Z, R(X,Y)]W - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W.$$

Keeping in mind that we only do cyclic sums over X, Y, Z. and that we have the Jacobi identity for operators:

$$\mathfrak{S}\, [\nabla_X, [\nabla_Y, \nabla_Z]] = 0.$$

we have

$$\mathfrak{S}(\nabla_X R)(Y,Z)W = \mathfrak{S}[\nabla_X, R(Y,Z)]W - \mathfrak{S}R(\nabla_X Y, Z)W - \mathfrak{S}R(Y, \nabla_X Z)W$$

$$= \mathfrak{S}[\nabla_X, [\nabla_Y, \nabla_Z]]W - \mathfrak{S}[\nabla_X, \nabla_{[Y,Z]}]W - \mathfrak{S}R(\nabla_X Y, Z)W - \mathfrak{S}R(Y, \nabla_X Z)W$$

$$\overset{\text{Jacobi identity + (1)}}{=} -\mathfrak{S}[\nabla_X, \nabla_{[Y,Z]}]W - \mathfrak{S}R(\nabla_X Y, Z)W + \mathfrak{S}R(\nabla_X Z, Y)W$$

$$= -\mathfrak{S}[\nabla_X, \nabla_{[Y,Z]}]W - \mathfrak{S}R(\nabla_X Y, Z)W + \mathfrak{S}R(\nabla_Y X, Z)W$$

$$\overset{\text{torsion-free}}{=} -\mathfrak{S}[\nabla_X, \nabla_{[Y,Z]}]W - \mathfrak{S}R([X,Y], Z)W$$

$$= -\mathfrak{S}[\nabla_X, \nabla_{[Y,Z]}]W - \mathfrak{S}[\nabla_{[X,Y]}, \nabla_Z]W + \mathfrak{S}\nabla_{[[X,Y],Z]}W$$

$$\overset{[X,Y]=-[Y,X]}{=} \mathfrak{S}[\nabla_{[Y,Z]}, \nabla_X]W - \mathfrak{S}[\nabla_{[X,Y]}, \nabla_Z]W$$

$$= 0. \qquad \square$$

In local coordinates, we write

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = R^k{}_{lij}\frac{\partial}{\partial x^k}.$$

$$= \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l}$$

$$= \nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^\gamma_{jl}\frac{\partial}{\partial x^\gamma}\right) - \nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^\gamma_{il}\frac{\partial}{\partial x^\gamma}\right)$$

$$= \frac{\partial \Gamma^\gamma_{jl}}{\partial x^i}\frac{\partial}{\partial x^\gamma} + \Gamma^\gamma_{jl}\Gamma^\mu_{i\gamma}\frac{\partial}{\partial x^\mu}$$

$$- \frac{\partial \Gamma^\gamma_{il}}{\partial x^j}\frac{\partial}{\partial x^\gamma} - \Gamma^\gamma_{il}\Gamma^\mu_{j\gamma}\frac{\partial}{\partial x^\mu}$$

$$= \left(\frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma} - \Gamma^\gamma_{il}\Gamma^k_{j\gamma}\right)\frac{\partial}{\partial x^k}.$$

That is,

$$\boxed{R^k{}_{lij} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma} - \Gamma^\gamma_{il}\Gamma^k_{j\gamma}}$$

We see $R^k{}_{lij} = -R^k{}_{lji}$, and $R^k{}_{lij} + R^k{}_{ijl} + R^k{}_{jli} = 0$.

§2.4 Riemannian curvature tensor.

Now we consider a Riemannian manifold (M,g) with a Levi-Civita connection ∇. We can use g to convert the $(1,3)$-tensor R to be a $(0,4)$-tensor:

$$\cancel{(X,Y,Z,W) \mapsto \mathbb{R}}$$

$$\langle R(X,Y)Z,W\rangle_g := R(W,Z,X,Y)$$

In local coordinates.

$$R_{klij} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right\rangle$$

$$= g_{km} R^m_{lij}.$$

$$= g_{km}\left(\frac{\partial \Gamma^m_{jl}}{\partial x^i} - \frac{\partial \Gamma^m_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^m_{i\gamma} - \Gamma^\gamma_{il}\Gamma^m_{j\gamma}\right)$$

$$g_{km}\frac{\partial \Gamma^m_{jl}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(g_{km}\Gamma^m_{jl}\right) - \Gamma^m_{jl}\frac{\partial g_{km}}{\partial x^i}$$

$$= \frac{1}{2}\frac{\partial}{\partial x^i}\left(g_{jk,l} + g_{kl,j} - g_{jl,k}\right) - \Gamma^m_{jl}\left(g_{mp}\Gamma^p_{ik} + g_{kp}\Gamma^p_{im}\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}\right) - g_{mp}\Gamma^m_{jl}\Gamma^p_{ik} - g_{kp}\Gamma^m_{jl}\Gamma^p_{im}$$

$$g_{km}\frac{\partial \Gamma^m_{il}}{\partial x^j}$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^j \partial x^i} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}\right) - g_{mp}\Gamma^m_{il}\Gamma^p_{jk} - g_{kp}\Gamma^m_{il}\Gamma^p_{jm}$$

$$\Rightarrow \boxed{\begin{aligned} R_{klij} = {} & \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k}\right) \\ & + g_{mp}\left(\Gamma^m_{il}\Gamma^p_{jk} - \Gamma^m_{jl}\Gamma^p_{ik}\right) \end{aligned}}$$

<u>Proposition 5</u>: We have the following identities.

(1) $\langle R(X,Y)Z, W\rangle = -\langle R(Y,X)Z, W\rangle$, i.e. $R_{klij} = -R_{klji}$

(2) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$ i.e. $R_{lkij} = -R_{lkij}$

(3) $\langle R(X,Y)Z, W\rangle + \langle R(Y,Z)X, W\rangle + \langle R(Z,X)Y, W\rangle = 0$

i.e. $R_{klij} + R_{kijl} + R_{kjli} = 0$

(4) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

i.e. $R_{klij} = R_{ijkl}$.

(5) $\nabla R(W,Z,X,Y,V) + \nabla R(W,Z,Y,V,X) + \nabla R(W,Z,V,X,Y) = 0$

<u>Proof</u>. (1) is obvious. (2) can be seen from it's local coordinates expression in.

One can also use the compatibility ~~of~~ of ∇ with g, to derive directly

$$\langle R(X,Y)Z, W\rangle = -\langle Z, R(X,Y)W\rangle.$$

(3) ~~is~~ follows directly from the First Bianchi Identity.

(5) follows from the second Bianchi Identity once we observe

$$\boxed{\nabla_V R(W,Z,X,Y) = \langle \nabla_V R(X,Y)Z, W\rangle}.$$

(4) is a consequence of properties (1) – (3).

Although one can also see (4) directly from its expression in local coordinates, it is deserved to have a look at the proof in [Spivak II, Chap 4D, Proposition 11]. A clever diagram proof taken from Milnor's <u>Morse Theory</u> book is presented there. □

There are interesting consequences ~~following~~ derived from those symmetries.

The Proposition 5 (1) (2), that is, $\langle R(X,Y)Z,W\rangle$ is ~~skew-symmetry~~ skew-symmetric in both (X,Y) and (Z,W), tells

Corollary 2. For two vector fields

$$(aX+bY,\ cX+dY) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

we have

$$\langle R(aX+bY, cX+dY)(cX+dY), cX+dY\rangle$$
$$= \left[\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^2 \langle R(X,Y)Y,X\rangle .$$

Proof. Exercise. □

$\underline{\text{Proposition 5 } \overbrace{(1)\ (2)}^{\Rightarrow (4)}\ (3)}$ tells the curvature tensor R is completely determined by the values of $\langle R(X,Y)Y,X\rangle$.

Corollary 3 Suppose $\langle R_1(X,Y)Y,X\rangle = \langle R_2(X,Y)Y,X\rangle, \forall X,Y$.

Then $\langle R_1(X,Y)Z,W\rangle = \langle R_2(X,Y)Z,W\rangle, \ \forall X,Y,Z,W.$

Proof: It is clearly suffice to prove that ~~a multilinear~~ if

$$\langle R(X,Y)Y,X\rangle = 0, \ \forall X,Y \Rightarrow \langle R(X,Y)Z,W\rangle = 0.$$

Now we have

$$0 = \langle R(X,Y+W)(Y+W),X\rangle = \overbrace{\langle R(X,Y)Y,X\rangle}^{0} + \langle R(X,Y)W,X\rangle$$
$$+\langle R(X,W)Y,X\rangle + \underbrace{\langle R(X,W)W,X\rangle}_{0}$$
$$\overset{(1)(2)(4)}{=} 2\langle R(X,Y)W,X\rangle, \quad \forall X,Y,W.$$

Moreover

$$0 = \langle R(X+Z,Y)W,X+Z\rangle = \overbrace{\langle R(X,Y)W,X\rangle}^{0} + \langle R(X,Y)W,Z\rangle$$
$$+\langle R(Z,Y)W,X\rangle + \underbrace{\langle R(Z,Y)W,Z\rangle}_{0}$$

$\cancel{= \langle R(X,Y)Z,W\rangle - \langle R(Z,Y)X,W\rangle}$

$$\Rightarrow \quad 0 = \langle R(X,Y)W, Z\rangle + \langle R(Z,Y)W, X\rangle$$

$$\overset{FBI}{=} -\langle R(Y,W)X, Z\rangle - \langle R(W,X)Y, Z\rangle + \langle R(Z,Y)W, X\rangle$$

$$\Rightarrow \quad \boxed{2\langle R(Z,Y)W, X\rangle = \langle R(Y,W)X, Z\rangle} \quad (1).$$

By a similar argument starting from

$$0 = \langle R(X+W, Y)Y, X+W\rangle$$

we will obtain

$$\boxed{2\langle R(X,Z)Y, W\rangle = \langle R(Y,Z)X, W\rangle} \quad (2).$$

Using symmetries, we can rewrite (1) and (2) as

$$2\langle R(Y,Z)X, W\rangle = \langle R(X,Z)Y, W\rangle$$

$$\text{and } 2\langle R(X,Z)Y, W\rangle = \langle R(Y,Z)X, W\rangle$$

which implies $\langle R(X,Z)Y, W\rangle = 0, \ \forall\ X,Y,Z,W.$ $\square$

Lecture 12, 2017.03.31.