$$\Rightarrow \quad 0 = \langle R(X,Y)W, Z\rangle + \langle R(Z,Y)W, X\rangle$$

$$\overset{\text{FBI}}{=} -\langle R(Y,W)X, Z\rangle - \langle R(W,X)Y, Z\rangle + \langle R(Z,Y)W, X\rangle$$

$$\Rightarrow \quad \boxed{2\langle R(Z,Y)W, X\rangle = \langle R(Y,W)X, Z\rangle} \quad (1).$$

By a similar argument starting from

$$0 = \langle R(X+W, Y)Y, X+W\rangle$$

we will obtain

$$\boxed{2\langle R(X,Z)Y, W\rangle = \langle R(Y,Z)X, W\rangle} \quad (2).$$

Using symmetries, we can rewrite (1) and (2) as

$$2\langle R(Y,Z)X, W\rangle = \langle R(X,Z)Y, W\rangle$$

$$\text{and } 2\langle R(X,Z)Y, W\rangle = \langle R(Y,Z)X, W\rangle$$

which implies $\langle R(X,Z)Y, W\rangle = 0$, $\forall\ X,Y,Z,W$. $\square$

Lecture 12 .2017.03.31.

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## §.3 Sectional Curvature

Consider another $(0,4)$-tensor: for $X,Y,Z,W \in \Gamma(TM)$.

$$G(X,Y,Z,W) = \langle X,Z\rangle_g \langle Y,W\rangle_g - \langle X,W\rangle_g \langle Y,Z\rangle_g$$

It is not hard to check $G$ satisfies the following properties

(1) $G(X,Y,Z,W) = -G(Y,X,Z,W)$

(2) $G(X,Y,W,Z) = -G(X,Y,Z,W)$

(3) $G(X,Y,Z,W) + G(Y,Z,X,W) + G(Z,X,Y,W) = 0$

~~Recall from~~ and

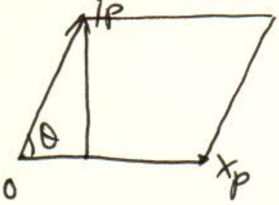
(4) $G(X,Y,Z,W) = G(Z,W,X,Y)$.

Recall from last section that (4) is actually a consequence of properties (1) − (3).

Hence $G$ behaves very similar to the Riemann curvature tensor $\langle R(X,Y)Z,W\rangle$.

In particular, For two linearly independent vector $X_p, Y_p \in T_pM$,

$$G(X_p, Y_p, X_p, Y_p) = \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2$$




$$= \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle\cos^2\theta$$

$$= \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle \sin^2\theta$$

equals the area of the parallelogram spanned by $X_p$ and $Y_p$.

By the proof of Corollary 2 (p.124), we have

$$G(aX_p + bY_p, cX_p + dY_p, aX_p + bY_p, cX_p + dY_p)$$
$$= \left[\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right]^2 G(X_p, Y_p, X_p, Y_p)$$

~~then~~ Therefore, we have

<u>Proposition 6</u>: The quantity

$$K(X_p, Y_p) := \frac{\langle R(X_p, Y_p)Y_p, X_p\rangle}{G(X_p, Y_p, X_p, Y_p)} = \frac{R(X_p, Y_p, X_p, Y_p)}{G(X_p, Y_p, X_p, Y_p)}$$
$$= \frac{R(X_p, Y_p, X_p, Y_p)}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2}$$

depends only on the two dimensional subspace

$$\Pi_p = \operatorname{span}(X_p, Y_p) \subset T_pM.$$

that is, it is independent of the choice of basis $\{X_p, Y_p\}$ of $\Pi_p$.

<u>Definition 1</u>: We will call (sectional curvature) $K(\Pi_p) = K(X_p, Y_p)$

the <u>sectional curvature</u> of $(M,g)$ at $p$ with respect to the plane $\Pi_p = \mathrm{span}(X_p, Y_p)$.

<u>Rmk</u>: (1) The sectional curvature $K$ is NOT a function on $M$ except when $\dim M = 2$.
(2) $K(Ag) = \frac{1}{A} K(g)$.

<u>Proposition 7</u>: Let $(M,g)$ be a 2-dim Rie. mfld, and let $X_p, Y_p \in T_pM$ be linearly independent. Then

$$K(p) = K(X_p, Y_p) = \frac{\langle R(X_p,Y_p)Y_p, X_p\rangle}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2}$$

is the same as the Gaussian curvature at $p$.

<u>Sketch of proof</u>: Let $(U, x^1, x^2)$ be a coordinate neighborhood of $p \in M$. By proposition 6, it suffices to ~~very~~ verify the proposition when

$$X_p = \frac{\partial}{\partial x^1}\Big|_p, \quad Y_p = \frac{\partial}{\partial x^2}\Big|_p$$

In this case

$$\left\langle R\left(\frac{\partial}{\partial x^1}\Big|_p, \frac{\partial}{\partial x^2}\Big|_p\right)\frac{\partial}{\partial x^2}\Big|_p, \frac{\partial}{\partial x^1}\Big|_p \right\rangle = R_{1212}(p)$$

while $G(X_p, Y_p, X_p, Y_p) = g_{11} g_{12} - g_{12}^2$.

Hence $$K(p) = \frac{R_{1212}(p)}{g_{11} g_{12} - g_{12}^2}.$$

Recall that the Gaussian curvature can be expressed via the first fundamental form

$$E\, dx^1 \otimes dx^1 + F\, dx^1 \otimes dx^2 + F\, dx^2 \otimes dx^1 + G\, dx^2 \otimes dx^2.$$

where in our case, $E = g_{11}$, $F = g_{12} = g_{21}$, $G = g_{22}$. $\square$

<u>Rmk</u>. Note that Propositions 6 and 7 together implies that Gaussian curvature is indeed independent of the choice of coordinates.

Or, equivalently, if $g_1, g_2$ are locally isometric, then they lead to the same Gauss curvature. This is the celebrated "Theorema Egregium". (高斯绝妙定理)

Rmk. We see in Exercise 5.(2) that isometries preserve Levi-Civita connections. That is, given $(M_1, g_1, \nabla^{(1)})$, $(M_2, g_2, \nabla^{(2)})$, and $\varphi: M_1 \to M_2$ be an isometry. Then for any $X, Y \in \Gamma(TM)$, we have

$$d\varphi(\nabla^{(1)}_X Y) = \nabla^{(2)}_{d\varphi(X)} d\varphi(Y)$$

As direct consequences, we see

if $R^{(i)}(X,Y)Z := \nabla^{(i)}_X \nabla^{(i)}_Y Z - \nabla^{(i)}_Y \nabla^{(i)}_X Z - \nabla^{(i)}_{[X,Y]} Z$,

then

$$d\varphi(R^{(1)}(X,Y)Z) = R^{(2)}(d\varphi(X), d\varphi(Y))\, d\varphi(Z)$$

$$\& \quad g_1(R^{(1)}(X,Y)Z, W) = g_2(R^{(2)}(d\varphi(X), d\varphi(Y))d\varphi(Z), d\varphi(W))\circ\varphi$$

~~Hence curvature tensor is~~ In particular, if $\varphi: M_1 \to M_2$ is an isometry s.t. $d\varphi(\Pi_p) = \Pi'_{\varphi(p)} \subseteq T_{\varphi(p)}M_2$.

We have the sectional curvature of $\Pi_p$ and that of $\Pi'_{\varphi(p)}$ are the same.

Proposition 8. Let $(M, g)$ be a Riemannian manifold, and let $\Pi_p$ be a 2-dim subspace of $T_pM$, spanned by $X_p, Y_p \in T_pM$. Let $\mathcal{O} \subset \Pi_p$ be a neighborhood of $0 \in T_pM$ on which $\exp_p$ is a diffeomorphism, let $i: \exp_p(\mathcal{O}) \to M$ be the inclusion, and let $\bar{R}$ be the ~~Riemannian~~ curvature tensor for $\exp_p(\mathcal{O})$ with the induced Riemannian metric $i^*g$. Then

$$\langle \bar{R}(X_p,Y_p)Y_p,X_p\rangle = \langle R(X_p,Y_p)Y_p,X_p\rangle$$

Consequently,
$$K(\Pi_p) = \frac{\langle R(X_p,Y_p)Y_p,X_p\rangle}{G(X_p,Y_p,X_p,Y_p)}$$
is the Gaussian curvature at $p$ of the surface $\exp_p(\mathcal{O})$. $\square$

Proposition 9. The Riemannian curvature tensor at $p$ is determined by all the sectional curvatures at $p$.

Proof: By Corollary 3. (p. 124) $\square$

Definition 2 A Riemannian manifold $(M,g)$ is said to have constant (sectional) curvature if its sectional curvature $K(\Pi_p)$ is a constant, i.e., is independent of $p$ and is independent of $\Pi_p \subset T_pM$.

Proposition 10. A Rie. mfld $(M,g)$ has constant curvature $k$ if and only if
$$\langle R(X,Y)W,Z\rangle = k\cdot G(X,Y,Z,W),\ \forall X,Y,Z,W \in \Gamma(TM)$$
i.e. $R(Z,W,X,Y) = R(X,Y,Z,W) = k\cdot G(X,Y,Z,W)$, or $\boxed{R = kG}$.

Proof: Recall both $\langle R(X,Y)Z,W\rangle$ and $G(X,Y,Z,W)$ satisfy the symmetries (1)–(3). (p. 123 & p. 125). Hence
$$S(X,Y,Z,W) := \langle R(X,Y)W,Z\rangle - kG(X,Y,Z,W)$$
satisfies (1) $S(X,Y,Z,W) = -S(Y,X,Z,W)$

(2) $S(X,Y,Z,W) = -S(X,Y,W,Z)$

(3) $S(X,Y,Z,W) + S(Y,Z,X,W) + S(Z,X,Y,W) = 0$

(4) $S(X,Y,Z,W) = S(Z,W,X,Y)$.

Notice our assumption implies $S(X,Y,X,Y) = 0$.

By the proof of Corollary 3, (p. 124), we have $S(X,Y,Z,W) = 0$ $\square$

Up to now, we haven't ~~use~~ made use of the second Bianchi ~~Ident~~ Identity. Prop 5 (5). (p. 123)

$$\nabla R(W,Z,X,Y,V) + \nabla R(W,Z,Y,V,X) + \nabla R(W,Z,V,X,Y) = 0.$$

In fact, it leads to the following Schur's theorem.

Theorem 2. (Schur) Let $(M,g)$ be a connected Riemannian manifold of dimension $n \geq 3$. If

$$K(\Pi_p) = f(p) \qquad (*)$$

depends only on $p$, then $(M,g)$ is of constant curvature.

Remark. (1). Thm 2 is obviously not true for $(M,g)$ with $\dim = 2$. We know in that case $(*)$ always holds, but $M$ need not be of constant curvature.

(2). Thm 2 says that the isotropy of a Rie. mfld, i.e. the property that at each point all directions are geometrically indistinguishable, implies the homogeneity, i.e., that all points are geometrically indistinguishable. In particular, a pointwise property implies a global one.

Before proving Thm 2, we prepare the following lemma.

Lemma 1. The tensor $G$ is parallel, i.e. $\nabla G = 0$.

Proof. For any $X, Y, Z, W, V \in \Gamma(TM)$, we have

$$\begin{aligned}
&(\nabla_V G)(X,Y,Z,W) \\
&= V(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle) \\
&\quad - \langle \nabla_V X, Z\rangle\langle Y,W\rangle - \langle X, \nabla_V Z\rangle\langle Y,W\rangle \\
&\quad - \langle X,Z\rangle\langle \nabla_V Y, W\rangle - \langle X,Z\rangle\langle Y, \nabla_V W\rangle
\end{aligned}$$

$- \langle \Phi \nabla_V X, W\rangle\langle Y,Z\rangle - \langle \Phi X, \nabla_V W\rangle\langle Y,Z\rangle$

$- \langle X,W\rangle\langle \nabla_V Y,Z\rangle - \langle X,W\rangle\langle Y,\nabla_V Z\rangle.$

By $V(\langle X,Z\rangle\langle Y,W\rangle) = V(\langle X,Z\rangle)\cdot\langle Y,W\rangle + \langle X,Z\rangle\cdot V\langle Y,W\rangle$

and compatibility of $\nabla$ with $g$, we conclude

$$(\nabla_V G)(X,Y,Z,W) = 0. \qquad \square$$

<u>Proof of Thm 2</u>. (An application of the second Bianchi Identity).

By assumption and Prop. 10, we have

$$R = f\cdot G. \quad \text{where } f: M \to \mathbb{R}$$

Lemma 1 above tells $\nabla G = 0$. Hence for all $V \in \Gamma(TM)$,

we have $\quad \nabla_V R = \nabla_V(f\cdot G) = V(f)G.$

By the second Bianchi Identity, we have

$$0 = \nabla_V R(W,Z,X,Y) + \nabla_X R(W,Z,Y,V) + \nabla_Y R(W,Z,V,X)$$

(*) $$= V(f)G(W,Z,X,Y) + X(f)G(W,Z,Y,V) + Y(f)G(W,Z,V,X).$$

for any $X,Y,Z,W,V \in \Gamma(TM)$.

Since ~~the RHS is~~ it is a tensor identity, the RHS only depends on $X_p, Y_p, Z_p, W_p, V_p \in T_pM$. Since $\dim(M) \geq 3$, we can pick $X_p, Y_p, V_p \in T_pM$ such that.

$$\cancel{X_p \neq 0} \ \langle X_p, Y_p\rangle = \langle X_p, V_p\rangle = \langle Y_p, V_p\rangle = 0$$

and $X_p \neq 0$, $Y_p \neq 0$, $|V_p| = 1$.

then (*) implies

$$0 = V_p(f)\left(\langle W_p,X_p\rangle\langle Z_p,Y_p\rangle - \langle W_p,Y_p\rangle\langle Z_p,X_p\rangle\right)$$
$$+ X_p(f)\left(\langle W_p,Y_p\rangle\langle Z_p,V_p\rangle - \langle W_p,V_p\rangle\langle Z_p,Y_p\rangle\right)$$
$$+ Y_p(f)\left(\langle W_p,V_p\rangle\langle Z_p,X_p\rangle - \langle W_p,X_p\rangle\langle Z_p,V_p\rangle\right)$$

Recall, we still have freedom for the choice of $W_p, Z_p$.

Let us set $Z_p = V_p$, then.

$$0 = X_p(f)\langle W_p, Y_p\rangle - Y_p(f)\langle W_p, X_p\rangle \quad \text{for any } W_p \in T_pM.$$

Hence $0 = X_p(f)Y_p - Y_p(f)X_p$.

However, $\langle X_p, Y_p\rangle = 0$. That is

$$X_p(f) = Y_p(f) = 0. \quad \forall\, X_p \neq 0,\ Y_p \neq 0.$$

So $f$ must be a constant function on $M$. $\square$.

## § 24 Ricci Curvature and Scalar curvature.

The <u>Ricci curvature tensor</u> is defined to be

$$Ric(Y, Z) := tr\,(X \mapsto R(X,Y)Z).$$

Notice that at $p$ $R(\cdot, Y)Z : T_pM \to T_pM$

is a linear map between vector spaces.

In local coordinate $(U, x^1, \cdots, x^n)$, we have

$$Ric_{pq} := Ric\left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^q}\right) = tr\left(X \mapsto R\left(X, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q}\right) = \sum_j R^j{}_{qjp}$$

Moreover, we have

$$\sum_j R^j{}_{qjp} = \sum_{i,j} g^{ij} g_{il} R^l{}_{qjp} = \sum_j g^{ij} R_{iqjp}$$

$$= \sum_{i,j} g^{ij}\left\langle R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q}, \frac{\partial}{\partial x^i}\right\rangle$$

$$= tr\left\langle R\left(\cdot, \frac{\partial}{\partial x^p}\right)\frac{\partial}{\partial x^q}, \cdot\right\rangle$$

$$= tr\, R\left(\cdot, \frac{\partial}{\partial x^q}, \cdot, \frac{\partial}{\partial x^p}\right)$$

Therefore

$$\mathrm{Ric}(Y,Z) = \mathrm{tr}\, \underbrace{R(\cdot, Y, \cdot, Z)}_{\text{as a } (0,2)\text{-tensor.}}$$

Recall the trace of a $(0,2)$-tensor from the section on Hessian.

In particular, we observe that

$$\mathrm{Ric}(Y,Z) = \mathrm{Ric}(Z,Y).$$

Recall from p. 103, if we denote for given $Y, Z$
$S(W,V) := R(W,Y,V,Z)$
and $g(\# S(W,\cdot), V) = g(R(V,Z)Y, W)$
$= g(R(Y,W)V, Z)$
$\Rightarrow \# S(W,\cdot) = R(W,Y)Z.$ $= g(R(W,Y)Z, V), \forall V$

That is, Ric is a symmetric $(0,2)$-~~tensor.~~ field on $M$.

Hence $\mathrm{tr}\, S := \mathrm{tr}(W \mapsto \# S(W,\cdot)) = \mathrm{tr}(W \mapsto R(W,Y)Z)$

Definition 3. (Ricci curvature) ~~For any unit~~ The Ricci curvature at $p$ in the direction $X_p \in T_pM$ is defined as

$$\mathrm{Ric}(X_p) := \mathrm{Ric}(X_p, X_p)$$

Remark: Ricci curvature is again NOT a function on $M$. We can think of the Ricci curvature as a function defined on one-dimensional subspaces of $T_pM$.

We can ask similar questions about Ricci curvature as in the case of sectional curvature: ~~Do~~ What information do we lose when restricting the Ricci curvature tensor to $\mathrm{Ric}(X,X)$? The answer is again that we don't lose anything.

Lemma 2. Let $T$ be a symmetric 2-tensor. Then for any $X, Y, Q$ we have

$$T(X,Y) = \frac{1}{2}\Big(T(X+Y, X+Y) - T(X,X) - T(Y,Y)\Big)$$

Hence

$$\mathrm{Ric}(X_p, Y_p) = \frac{1}{2}\big(\mathrm{Ric}(X_p + Y_p) - \mathrm{Ric}(X_p) - \mathrm{Ric}(Y_p)\big).$$

We actually should have normalized the lengths of the vector, along which the Ricci curvature is calculated.

After that, Ricci curvature is defined on "tangent directions".

$$Ric\left(\frac{X}{\|X\|}\right) =: \frac{Ric(X)}{g(X,X)} = \frac{Ric(X,X)}{g(X,X)}$$

Definition 4. The Rie. mfld is called an Einstein manifold with Einstein constant $k$, if

$$Ric(X) = k\, g(X,X), \quad \forall X \in \Gamma(TM).$$

i.e. $M$ has "constant Ricci curvature".

Remark: Let $X_p \in T_pM$ be an unit tangent vector. Extend it to be an othornomal basis $\{X_p, e_2, \cdots, e_n\}$ of $T_pM$. Then

$$\begin{aligned} Ric(X_p) &= tr\, R(\cdot, X_p, \cdot, X_p) \\ &= \sum_{i=2}^{n} R(e_i, X_p, e_i, X_p) \\ &= \sum_{i=2}^{n} K(e_i, X_p). \end{aligned}$$

In particular, If $(M,g)$ has constant curvature $k$, then $(M,g)$ is Einstein with Einstein constant $(n-1)k$.

Proposition 11. A Rie. mfld is an Einstein manifold with Einstein constant $k$ if and only if

$$Ric = kg.$$

Proof: Define $T(X,Y) = Ric(X,Y) - k g(X,Y)$. Hence $T$ symmetric. By assumption $T(X,X) = 0$.

Lemma 2 tells $T(X,Y) = 0$, i.e. $Ric = kg$. □

Lecture 13, 2017. 04. 06.

We also have the following version of Schur's theorem.

Theorem 3. (Schur). Let $(M,g)$ be a connected Riemannian manifold (135)

of dimension $\geq 3$. If $Ric(X_p) = f(p)\, g(X_p, X_p)$, $\forall X_p \in T_pM$, where $f(p)$ depends only on $p$, then $(M,g)$ is Einstein.

Proof: Apply the second Bianchi identity in the same manner as in Theorem 2. (p.130).

Step 1. By proposition 11, the assumptions implies

$$Ric = f g.$$

Note for Levi-Civita connection, we have automatically

$$\nabla g = 0$$

Hence $\forall\ V \in \Gamma(TM)$, we have

$$\nabla_V Ric = V(f)\, g.$$

Step 2. Apply 2nd Bianchi Identity. At $p \in M$, pick an normal coordinate $(U, x^1, \dots, x^n)$, we have for $X_p, Y_p, V_p \in T_pM$,

$$\begin{aligned}
\nabla_V Ric(X_p, Y_p) &= V_p(Ric(X_p, Y_p)) - Ric(\nabla_{V_p} X_p, Y_p) - Ric(X_p, \nabla_{V_p} Y_p) \\
&= V_p\Big(\sum_{i=1}^n R\big(\tfrac{\partial}{\partial x^i}, X_p, \tfrac{\partial}{\partial x^i}, Y_p\big)\Big) \\
&\quad - \sum_{i=1}^n R\big(\tfrac{\partial}{\partial x^i}, \nabla_{V_p} X_p, \tfrac{\partial}{\partial x^i}, Y_p\big) \\
&\quad - \sum_{i=1}^n R\big(\tfrac{\partial}{\partial x^i}, X_p, \tfrac{\partial}{\partial x^i}, \nabla_{V_p} Y_p\big) \\
&= \sum_{i=1}^n \Big( (\nabla_{V_p} R)\big(\tfrac{\partial}{\partial x^i}, X_p, \tfrac{\partial}{\partial x^i}, Y_p\big)\Big)
\end{aligned}$$

(we used $\nabla_{V_p} \frac{\partial}{\partial x^i} = 0$ since normal coord.)

2nd Bianchi Identity implies

$$\begin{aligned}
0 &= \sum_{i=1}^n \Big[ \nabla_{V_p} R\big(\tfrac{\partial}{\partial x^i}, X_p, \tfrac{\partial}{\partial x^i}, Y_p\big) + \nabla_{\frac{\partial}{\partial x^i}} R\big(\tfrac{\partial}{\partial x^i}, X_p, Y_p, V_p\big) + \nabla_{Y_p}\big(\tfrac{\partial}{\partial x^i}, X_p, V_p, \tfrac{\partial}{\partial x^i}\big)\Big] \\
&= \nabla_{V_p} Ric(X_p, Y_p) - \nabla_{Y_p} Ric(X_p, V_p) + \sum_{i=1}^n \nabla_{\frac{\partial}{\partial x^i}} R\big(\tfrac{\partial}{\partial x^i}, X_p, Y_p, V_p\big) \\
&= V_p(f)\, g(X_p, Y_p) - Y_p(f)\, g(X_p, V_p) + \sum_{i=1}^n \nabla_{\frac{\partial}{\partial x^i}} R\big(\tfrac{\partial}{\partial x^i}, X_p, Y_p, V_p\big).
\end{aligned}$$

Step 3. Pick special $X_p, Y_p, V_p$.

Let $X_p = Y_p = \frac{\partial}{\partial x^j}$, $V_p = \frac{\partial}{\partial x^h}$. ~~for so~~

we have

$$0 = \frac{\partial f}{\partial x^h} - \frac{\partial f}{\partial x^i}\delta_{jh} + \sum_{i=1}^{n} \nabla_{\frac{\partial}{\partial x^i}} R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h}\right)$$

Summing $j$ from $1$ to $n$,

$$0 = n\cdot\frac{\partial f}{\partial x^h} - \frac{\partial f}{\partial x^h}\cancel{\phi} - \sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_{\frac{\partial}{\partial x^i}} R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^h}\right)$$

$$= (n-1)\frac{\partial f}{\partial x^h} - \sum_{i=1}^{n}\left(\nabla_{\frac{\partial}{\partial x^i}} Ric\right)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^h}\right)$$

$$= (n-1)\frac{\partial f}{\partial x^h} - \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\cdot\underbrace{g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^h}\right)}_{\delta_{ih}}$$

$$= (n-2)\frac{\partial f}{\partial x^h}.$$

Hence, when $n \geq 3$, we have $\frac{\partial f}{\partial x^h} = 0$, $\forall\, h = 1, \dots, n$.

This implies that $f \equiv \text{const}$. □

Remark (1): In fact, Thm 3 implies Thm 2. (p. 130). Notice that

$K(\Pi_p) = f(p)$ depends only on $p$ implies

$$\frac{Ric(X_p)}{g(X_p, X_p)} = f(p) \text{ depends only on } p. \xRightarrow{\text{Thm 3}} \frac{Ric(X_p)}{g(X_p, X_p)} = \text{const.} \quad ①$$

$K(\Pi_p) = f(p)$ depends only on $p$ implies $\frac{Ric(X_p)}{g(X_p, X_p)} = (n-1)f(p)$ ②

① + ② $\Rightarrow$ $K(\Pi_p) = f(p) \equiv \text{const}$.

(2) In [BSSG] (白正国), $(M, g)$ is ~~defined~~ called Einstein if

$$Ric(X_p) = f(p)\, g(X_p, X_p)$$

where $f(p)$ depends only on $p$. By Thm 3, there is no difference from our definition in case $\dim \geq 3$.

[BSSG] (白正国)'s notation has the property that "any 2-dim Rie. mfld is Einstein".

<u>Definition 5</u> (Scalar curvature) The <u>scalar curvature</u> $S$ is defined as the trace of the Ricci curvature tensor (which is a symmetric $(0,2)$-tensor), i.e.

$$S = g^{ij} Ric_{ij} = \operatorname{tr} Ric(\cdot,\cdot)$$

Remark (1) $S$ is indeed a function on $M$.

(2) Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_pM$, we have

$$\begin{aligned} S(p) = \operatorname{tr}(Ric)(p) &= \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i=1}^{n} Ric(e_i) \\ &= \sum_{i=1}^{n} \operatorname{tr} R(\cdot, e_i, \cdot, e_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} R(e_j, e_i, e_j, e_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} K(e_j, e_i) \\ &= 2 \sum_{i<j} K(e_i, e_j) \end{aligned}$$

(3) If $(M,g)$ is of constant curvature $k$, we have

$$Ric = (n-1)k\, g, \quad \text{and } S = n(n-1)k.$$

If $(M,g)$ is Einstein with Einstein constant $k$, we have

$$S = nk.$$

<u>Proposition 12</u>: An $n(\geq 3)$-dimensional Riemannian manifold $(M,g)$ is Einstein iff

$$Ric = \frac{S}{n} g.$$

<u>Proof</u>: $\Rightarrow$ ~~Obvi.~~ By definition.

$\Leftarrow$ $Ric = \frac{S}{n} g$ where $\frac{S}{n}(p)$ depends only on $p$.

Schur's thm $\underset{n\geq 3}{\Rightarrow}$ $\frac{S}{n} \equiv \text{const}$. $\square$

In principle, Ricci curvature provides less information than sectional curvature, and scalar curvature provides even less information than

Ricci curvature. But in dimension 2 or 3, something special happens.

$n=2$ : $K(\Pi_p) = \frac{Ric(X_p)}{g(X_p, X_p)} = 2S(p).$

~~nes~~. There is no difference from an information point of view in knowing $K$, $Ric$, or $S$.

$n=3$: There is no difference in knowing $K$ ~~and~~ or $Ric$:

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $T_pM$, then

$$\begin{cases} K(e_1,e_2) + K(e_1,e_3) = Ric(e_1) \\ K(e_1,e_2) + K(e_2,e_3) = Ric(e_2) \\ K(e_1,e_3) + K(e_2,e_3) = Ric(e_3) \end{cases}$$

In other words,

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} K(e_1,e_2) \\ K(e_2,e_3) \\ K(e_1,e_3) \end{pmatrix} = \begin{pmatrix} Ric(e_1) \\ Ric(e_2) \\ Ric(e_3) \end{pmatrix} \qquad (**)$$

Notice that $\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2 \neq 0.$

Therefore any sectional curvature can be computed from $Ric$.

<u>Proposition 13</u>. $(M^3, g)$ is <u>Einstein</u> iff $(M^3, g)$ has <u>constant sectional curvature</u>.

<u>Proof</u>: $\Leftarrow$ By definition

$\Rightarrow$. Souling $(**)$ for the case $Ric(e_1) = Ric(e_2) = Ric(e_3)$ we obtain $K(e_1,e_2) = K(e_2,e_3) = K(e_1,e_3)$. $\square$.

But for scalar curvature: when $n=3$, there are metrics with constant scalar curvature that are not Einstein.

We will see whether the (sectional, Ricci, scalar) curvatures of Riemannian mflds are constant, or more generally although not

not constant but still bounded by some inequalities have much implications to the analysis, geometry and topology of $(M,g)$.

In particular, we explain the terminology "$\underline{\underline{Ric \geq k}}$", (Ricci curvature is lower bounded), this means more precisely that

$$Ric(X_p) = \underset{(0,2)\text{-tensor}}{Ric}(X_p, X_p) \geq k\, \underset{(0,2)\text{-tensor}}{g}(X, X), \quad \forall X.$$

We have discussed several times that a symmetric $(0,2)$-tensor has a "corresponding" linear transformation. Since

$$\begin{aligned} Ric(X_p, X_p) &= \sum_i R(e_i, X_p, e_i, X_p) \\ &= \sum_i \langle R(e_i, X_p)X_p, e_i\rangle \\ &= \sum_i \langle R(X_p, e_i)e_i, X_p\rangle \\ &= \langle \# Ric(X_p, \cdot), X_p\rangle \end{aligned}$$

$$\Rightarrow \# Ric(X_p, \cdot) = \sum_i R(X_p, e_i)e_i.$$

$\Rightarrow X_p \mapsto \sum_i R(X_p, e_i)e_i$ is a linear transformation between $T_pM$ and $T_pM$. The condition "$Ric \geq k$" is equivalent to say ~~the~~ all eigenvalues of $X_p \mapsto \sum_i R(X_p, e_i)e_i$ are $\geq k$.

Let us mention the following theorem of Lohkamp.

Theorem (Lohkamp, Annals of Math. 140 (1994), 655–683) For each manifold $M^n$, $n \geq 3$, there is a complete metric $g_M$ with

$$-a(n)\, g_M < Ric(g_M) < -b(n)\, g_M.$$

with constants $a(n) > b(n) > 0$ depending only on the dimension $n$.

<u>Theorem</u> (Lohkamp) For each manifold $M^n$, $n \geq 3$, there is a complete metric $g_M$ with negative Ricci curvature and finite volume.

That is, there are $\underline{\underline{\text{no}}}$ topological obstructions for negative Ricci curvature metrics.