补 2001

命题1.1.（等距变换等价描述）设两个参数表示分别为
$r = r(u,v)$ 和 $\tilde{r} = \tilde{r}(\tilde{u},\tilde{v})$
的曲面 M 和 $\tilde{M}$ 之间存在 1:1 对应 $\sigma: r(u,v) \mapsto \tilde{r}(\tilde{u},\tilde{v})$
则 (i) σ 是等距变换当且仅当 它们的第一基本形式在对应 σ 下满足

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = J_\sigma \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J_\sigma^T, \quad \text{其中 } J_\sigma = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$

(ii) σ 是等距变换当且仅当 $\forall\, p \in M$, $\forall\, v, w \in T_pM$ 有 $\langle \sigma_*(v), \sigma_*(w) \rangle = \langle v, w \rangle$.

证明：σ 等距变换 $\Leftrightarrow$ $\forall\, p \in M$, $\forall\, v \in T_pM$ 有

$$I(v,v) = \tilde{I}(\sigma_*(v), \sigma_*(v)) \qquad (1)$$

即 $\langle v, v \rangle = \langle \sigma_*(v), \sigma_*(v) \rangle$

因为 $\{r_u, r_v\}$ 是 T_pM 一组基，故可设

$$v = a r_u + b r_v$$

$$\Rightarrow I(v,v) = (a \;\; b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned}
\sigma_*(v) &= a\sigma_*(r_u) + b\sigma_*(r_v) \\
&= a\left(\tilde{r}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial u} + \tilde{r}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial u}\right) + b\left(\tilde{r}_{\tilde{u}} \frac{\partial \tilde{u}}{\partial v} + \tilde{r}_{\tilde{v}} \frac{\partial \tilde{v}}{\partial v}\right) \\
&= \left(a\frac{\partial \tilde{u}}{\partial u} + b\frac{\partial \tilde{u}}{\partial v}\right)\tilde{r}_{\tilde{u}} + \left(a\frac{\partial \tilde{v}}{\partial u} + b\frac{\partial \tilde{v}}{\partial v}\right)\tilde{r}_{\tilde{v}}
\end{aligned}$$

$$\Rightarrow \tilde{I}(\sigma_*(v), \sigma_*(v)) = \left(a\frac{\partial \tilde{u}}{\partial u} + b\frac{\partial \tilde{u}}{\partial v},\; a\frac{\partial \tilde{v}}{\partial u} + b\frac{\partial \tilde{v}}{\partial v}\right) \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} a\frac{\partial \tilde{u}}{\partial u} + b\frac{\partial \tilde{u}}{\partial v} \\ a\frac{\partial \tilde{v}}{\partial u} + b\frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$

$$= (a \;\; b) \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

故而 (1) 意味着

$$(a\ b)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (a,b)\, J_\sigma \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J_\sigma^T \begin{pmatrix} a \\ b \end{pmatrix},\ \forall (a,b)$$

由二次型和对称阵的相互决定性，得

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = J_\sigma \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J_\sigma^T. \qquad (2)$$

故而 $\forall\ v, w \in T_pM$，记 $v = a r_u + b r_v$，
$w = c r_u + d r_v$

有 $$\langle v, w\rangle = (a\ b)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\overset{(2)}{=} (a,b)\, J_\sigma \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} J_\sigma^T \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \langle \sigma_*(v), \sigma_*(w)\rangle.$$

□