RIEMANNIAN GEOMETRY
EXCERCISE 1

1. (i) An immersion $f : N \to \mathbb{R}^{n+1}$ of an n dimensional smooth manifold into $\mathbb{R}^{n+1}$ is called a hypersurface. Suppose f can be expressed locally in a coordinate neighborhood $(U, u^1, \ldots, u^n)$ as

$$x^k = f^k(u^1, \ldots, u^n), \;\; 1 \leq k \leq n+1,$$

where $(x^1, \ldots, x^{n+1})$ are the coordinates in $\mathbb{R}^{n+1}$. Let g_0 be the standard Eulidean metric on $\mathbb{R}^{n+1}$. Show

$$(f^* g_0)_{|U} = \sum_{k,i,j} \frac{\partial f^k}{\partial u^i} \frac{\partial f^k}{\partial u^j} du^i \otimes du^j.$$

(ii) Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Consider the coordinate neighborhood $(U, y^1, \ldots, y^n)$ where

$$U := \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : x^{n+1} \neq 1\},$$

and

$$(y^1, \ldots, y^n) := \left(\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}} \right).$$

Prove the induced metric on S^n in $(U, y^1, \ldots, y^n)$ is

$$\frac{4}{(1 + |y|^2)^2} \sum_{i=1}^{n} dy^i \otimes dy^i.$$

2. (i) On $\mathbb{R}^2 \setminus \{\text{half line}\}$ we have polar coordinates (r, θ). In these coordinates, compute the Riemannian metric on it induced from the Euclidean metric on $\mathbb{R}^2$.

(ii) A *surface of revolution* consists of a profile curve

$$c(t) = (r(t), 0, z(t)) : I \to \mathbb{R}^3,$$

where $I \subset \mathbb{R}$ is open and $r(t) > 0$ for all t. By rotating this curve around the z-axis, we get a surface that can be represented as

$$(t, \theta) \to f(t, \theta) = (r(t) \cos\theta, r(t) \sin\theta, z(t)).$$

Suppose the curve $c(t)$ is parametrized by arc length. Compute the Riemannain metric of this surface, in the above coordinate, from the Euclidean metric on $\mathbb{R}^2$.

(iii) The unit sphere $S^2 \subset \mathbb{R}^3$ can be though of as a surface of revolution by revolving

$$t \to R\left(\sin\left(\frac{t}{R}\right), 0, \cos\left(\frac{t}{R}\right) \right).$$

Use (ii) to derive the induced metric.

(iv)Consider the surface of revolution produced by the profile curve

$$t \to R\left(\sinh\left(\frac{t}{R}\right), 0, \cosh\left(\frac{t}{R}\right) \right).$$

Compute the metric induced from $(\mathbb{R}^3, dx \otimes dx + dy \otimes dy - dz \otimes dz)$.

3. (Christoffel symbols) Let $(U, x = (x^1, \dots, x^n))$ be a chart of a Riemannian manifold M. Let

$$(x^1, \dots, x^n) \to (y^1, \dots, y^n) \tag{0.1}$$

be a smooth coordinate change, and the Riemannian metric can be written as

$$g_{ij}(x) dx^i \otimes dx^j \text{ and } h_{\alpha\beta}(y) dy^\alpha \otimes dy^\beta$$

respectively.

(i) Show the transformation formula of g^{ij} under the coordinate change (0.1) is

$$g^{ij}(x) = h^{\alpha\beta}(y(x)) \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}.$$

(ii) Compute the transformation formulae of the Christoffel symbols Γ^i_{jk} under the coordinate change (0.1). Do they define a tensor?

(iii) Let $\gamma : [a, b] \to U$ be a smooth curve. Denote $\dot{x}^i(t) := \frac{d}{dt} x^i(\gamma(t))$. Compute the transformation formula of

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \dot{x}^j(t) \dot{x}^k(t)$$

under the coordinate change (0.1).