RIEMANNIAN GEOMETRY
EXCERCISE 3

1. Let M be a smooth manifold. Find a (nontrivial) affine connection on M via using "partition of unity".

2. Let M be a smooth manifold with an affine connection ∇. Let $X, Y \in \Gamma(TM)$. Let $U \subset M$ be an open subset. Prove that if $Y_{|_U} = 0$, then $(\nabla_X Y)_{|_U} = 0$.

3. (Covariant derivatives of tensor fields via parallel transport) Recall that for an isomorphism $\varphi : V \to W$ between two vector spaces V and W, there is an adjoint isomorphism

$$\varphi^* : W^* \to V^*,$$

between their dual spaces. For $\alpha \in W^*$, we have

$$\varphi(\alpha)(v) := \alpha(\varphi(v)), \ \ \forall\ v \in V.$$

Then, for any $v_i \in V,\ \alpha^j \in V^*$, we define

$$\widetilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s) = \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s).$$

By linearity, we can extend $\widetilde{\varphi}$ to be defined on all (r, s)-tensor, $\otimes^{r,s}V$, over V. This defines an isomorphism

$$\widetilde{\varphi} : \otimes^{r,s}V \to \otimes^{r,s}W.$$

Let M be a smooth manifold with an affine connection ∇. Let $c : I \to M$ be a smooth curve in M with $c(0) = p \in M$ and $\dot{c}(0) = X_p \in T_pM$. Recall that the parallel transport

$$P_{c,t} : T_{c(0)}M \to T_{c(t)}M,$$

is an isomorphism. As described above, we can extend it to be an isomorphism

$$\widetilde{P}_{c,t} : \otimes^{r,s}T_{c(0)}M \to \otimes^{r,s}T_{c(t)}M.$$

For any $A \in \Gamma(\otimes^{r,s}TM)$, we define

$$\nabla_{X_p} A := \lim_{h \to 0} \frac{1}{h} \left(\widetilde{P}^{-1}_{c,h} A(c(h)) - A(p) \right).$$

Let $Y \in \Gamma(TM), w, \eta \in \Gamma(T^*M)$. Consider the $(1, 2)$-tensor filed $K := Y \otimes w \otimes \eta$.

(i) Show that

$$\nabla_{X_p} K = \nabla_{X_p} Y \otimes w \otimes \eta + Y \otimes \nabla_{X_p} w \otimes \eta + Y \otimes w \otimes \nabla_{X_p} \eta.$$

(ii) Let $C : \Gamma(\otimes^{1,2}TM) \to \Gamma(\otimes^{0,1}TM)$ be the contraction map that pairs the first vector with the first covector. For example, $CK = w(Y)\eta$. Show that

$$\nabla_{X_p}(CK) = C(\nabla_{X_p} K).$$