RIEMANNIAN GEOMETRY
EXCERCISE 4

1. (Induced connection) Let M, N be two smooth manifold and $\varphi : N \to M$ be a smooth map. A vector field along φ is an assignment

$$x \in N \mapsto T_{\varphi(x)}M.$$

Let $\{E_i\}_{i=1}^n$ be a frame field in a neighborhood U of $\varphi(x) \in M$. Then for any $y \in \varphi^{-1}(U)$, we have

$$V(x) = V^i(x)E_i(\varphi(x)).$$

Let $u \in T_xN$. We define

$$\widetilde{\nabla}_u V := u(V^i)(x)E_i(\varphi(x)) + V^i(x)\nabla_{d\varphi(u)}E_i, \tag{0.1}$$

where ∇ is an affine connection on M.

(i) Check that $\widetilde{\nabla}_u V$ is well defined, i.e., (**??**) is independent of the choices of $\{E_i\}$.

(ii) Let g be a Riemannian metric on M. Prove that if ∇ on M is compatible with g, then for vector fields V, W along φ, and $u \in T_xN$, we have

$$u\langle V, W\rangle = \langle \widetilde{\nabla}_u V, W\rangle + \langle V, \widetilde{\nabla}_u W\rangle.$$

(iii) Prove that if ∇ on M is torsion free, then for any $X, Y \in \Gamma(TN$, we have

$$\widetilde{\nabla}_X d\varphi(Y) - \widetilde{\nabla}_Y d\varphi(X) - d\varphi\left([X, y]\right) = 0.$$

2. Let S^n be the sphere with the induced metric g from the Euclidean metric in $\mathbb{R}^{n+1}$. We denote by $\overline{\nabla}$ the canonical Levi-Civita connection on $\mathbb{R}^{n+1}$. For any $X, Y \in \Gamma(T\mathbb{S}^n)$, one can extend X, Y to smooth vector field $\overline{X}, \overline{Y}$ on $\mathbb{R}^{n+1}$, at least near $\mathbb{S}^n$.

By locality, the vector $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any $p \in \mathbb{S}^n$ depends only on $\overline{X}(p) = X(p)$ and the vectors $\overline{Y}(q) = Y(q)$ for $q \in \mathbb{S}^n$. That is, $\overline{\nabla}_{\overline{X}}\overline{Y}$ is independent of the extension of X, Y we choose. So we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\overline{X}}\overline{Y}$ at points on $\mathbb{S}^n$.

We define $\nabla_X Y$ to be the orthogonal projection of $\overline{\nabla}_X Y$ onto the tangent space of $\mathbb{S}^n$, i.e.,

$$\nabla_X Y := \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathbf{n}\rangle \mathbf{n},$$

where $\mathbf{n}$ is the unit out normal vector on $\mathbb{S}^n$.

(i) Prove that ∇ is an affine connection on $\mathbb{S}^n$.

(ii) Prove that ∇ is the Levi-Civita connection of $(\mathbb{S}^n, g)$.

3. Let N_1, N_2 be two submanifolds of a complete Riemannian manifold (M, g) without boundary, and let $\gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) \in N_1$, $\gamma(a) \in N_2$ and γ is the shortest curve from N_1 to N_2. Prove that $\dot\gamma(0)$ is perpendicular to $T_{\gamma(0)}N_1$, and $\dot\gamma(a)$ is perpendicular to $T_{\gamma(t)}N_2$. (Hint: Use the First Variation Formula.)

4. Let $c : [0, a] \to M$ be a piecewise smooth curve. That is, there exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = a$$

such that c is smooth on each interval $[t_i, t_{i+1}]$.

(i) At the break points t_i, there are two possible values for the velocity vector filed along c: a right derivative and a left derivative:

$$\dot{c}(t_i^+) = \frac{dc}{dt}_{|_{[t_i, t_{i+1}]}}(t_i),\ \dot{c}(t_i^-) = \frac{dc}{dt}_{|_{[t_{i-1}, t_i]}}(t_i).$$

Let $F : [0, a] \times (-\epsilon, \epsilon) \to M$ be a piecewise smooth variation of c. Derive the First Variation Formula of the energy functional. (Hint: Make use of the formula for smooth curves we discussed during the course.)

(ii) Let $V(t)$ be a piecewise smooth vector filed along the curve c. Show that there exists a variation $F : [0, a] \times (-\epsilon, \epsilon) \to M$ such that $V(t)$ is the variational field of F; in addition, If $V(0) = V(a) = 0$, it is possible to choose F as a proper variation. (Hint: Use exponential maps.)

(iii) (Characterization of geodesics) Prove that a piecewise smooth curve $c : [0, a] \to M$ is a geodesic if and only if, for every proper variation F of c, we have

$$E'(0) = 0.$$