RIEMANNIAN GEOMETRY
EXCERCISE 5

Recall for $X, Y, Z \in \Gamma(TM)$, the curvature tensor is defined as

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

1. (Ricci Identity)
   (i) Let $X, Y \in \Gamma(TM)$, $\phi \in \Gamma(\otimes^{r,s} TM)$. Prove that

$$\begin{aligned} &\nabla^2 \phi(w_1, \ldots, w_r, X_1, \ldots, X_s, X, Y) - \nabla^2 \phi(w_1, \ldots, w_r, X_1, \ldots, X_s, Y, X) \\ =& - R(X,Y)\phi(w_1, \ldots, w_r, X_1, \ldots, X_s), \end{aligned}$$

   for any $w_1, \ldots, w_r \in \Gamma(T^*M), X_1, \ldots, X_s \in \Gamma(TM)$.
   (ii) Let $Z \in \Gamma(TM)$. Derive from Ricci Identity the following identity in local coordinate $(U, x^1, \ldots, x^n)$:

$$Z^\ell{}_{,jk} - Z^\ell{}_{,kj} = -Z^i R^\ell{}_{ijk},$$

   where $R^\ell{}_{ijk} \frac{\partial}{\partial x^\ell} := R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) \frac{\partial}{\partial x^i}$, and $\nabla^2 Z = Z^\ell{}_{,jk} \frac{\partial}{\partial x^\ell} \otimes dx^j \otimes dx^k$.

2. (Curvature tensor R is determined by the values $\langle R(X,Y)Y, X\rangle$)
   (i) Show that if $\langle R(X,Y)Y, X\rangle = 0$ for any $X, Y \in \Gamma(TM)$, then $\langle R(X,Y)W, X\rangle = 0$ for any $X, Y, W \in \Gamma(TM)$.
   (ii) Show that if $\langle R(X,Y)W, X\rangle = 0$ for any $X, Y, W \in \Gamma(TM)$, then

$$2\,\langle R(Y,Z)X, W\rangle = \langle R(X,Z)Y, W\rangle\,,$$

   for any $X, Y, Z, W \in \Gamma(TM)$. (Hint: using the first Bianchi identity.)
   (iii) Show that if $2\,\langle R(Y,Z)X, W\rangle = \langle R(X,Z)Y, W\rangle$, $\forall X, Y, Z, W \in \Gamma(TM)$, then

$$\langle R(X,Y)Z, W\rangle = 0,$$

   for any $X, Y, Z, W \in \Gamma(TM)$.

3.(An alternative proof of 5)
Prove that for any $X, Y, Z, W \in \Gamma(TM)$, we have

$$\begin{aligned} -6\,\langle R(X,Y)Z, W\rangle = \frac{\partial^2}{\partial s \partial t}_{|s=t=0} &[\,\langle R(X+sZ, Y+tW)(Y+tW), X+sZ\rangle \\ &- \langle R(X+sW, Y+tZ)(Y+tZ), X+sW\rangle] \end{aligned}$$

(Hint: using the first Bianchi identity.)

4. (The Second Variation Formula for length) Let $\gamma : [a,b] \to M$ be a smooth curve and

$$F : [a,b] \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$$

be a 2-parameter variation of γ. Denote by

$$V(t) := \frac{\partial F}{\partial v}(t, 0, 0), \quad W(t) = \frac{\partial F}{\partial w}(t, 0, 0)$$

the two corresponding variational fields. Let $L(v,w) := L(\gamma_{v,w})$ be the length of the curve $\gamma_{v,w}(t) := F(t, v, w), t \in [a,b]$.

(1) Show that

$$\frac{\partial}{\partial w \partial v} L(v,w) = \int_a^b \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|} \Bigg\{ \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle - \left\langle R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right) \frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle + \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle - \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|^2} \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle \Bigg\} dt,$$

where $\left\| \frac{\partial F}{\partial t} \right\| := \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle^{\frac{1}{2}}$.

(2) Let γ be a normal geodesic. Show that

$$\frac{\partial}{\partial w \partial v}_{|v=w=0} L(v,w) = \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V,T \rangle T \langle W,T \rangle \right) dt + \langle \nabla_W V, T \rangle |_a^b,$$

where $T(t) := \dot{\gamma}(t)$ is the velocity field along γ.

(3) Consider the orthogonal component $V^{\perp}, W^{\perp}$ of V, W with respect to T, that is

$$V^{\perp} := V - \langle V, T \rangle T,$$
$$W^{\perp} := W - \langle W, T \rangle T.$$

Show that

$$\frac{\partial}{\partial w \partial v}_{|v=w=0} L(v,w) = \int_a^b \left(\langle \nabla_T V^{\perp}, \nabla_T W^{\perp} \rangle - \langle R(W^{\perp}, T)T, V^{\perp} \rangle \right) dt + \langle \nabla_W V, T \rangle |_a^b,$$