<u>Proof</u>: ([PP, Prop. 5.5.1] [doC, 3.7 Theorem])

This is a refinement of Theorem 2. (p.36) The proof again needs unraveling tangent spaces and identifications. The tangent space $T_{(p,p)}(M\times M)$ is naturally identified with $T_pM \times T_pM$. The tangent space $T_{(p,0_p)}(TM)$ is naturally identified with $T_pM \times T_{0_p}(T_pM) \cong T_pM \times T_pM$.

For the map $F : \mathcal{U} \subset TM \to M\times M$, its differential $dF(p,0_p)$ at $(p,0_p)\in TM$ can be considered a linear map

$$dF(p,0_p) : T_pM \times T_pM \to T_pM \times T_pM.$$

Thus the map $dF(p,0_p)$ can be represented as a $2n\times 2n$ matrix.

① For any $v \in T_pM$, consider the curve in TM:

$$(\gamma_{p,v}(t),\ 0_{\gamma_{p,v}(t)}).$$

and we have $F(\gamma_{p,v}(t), 0_{\gamma_{p,v}}(t)) = (\gamma_{p,v}(t), \gamma_{p,v}(t))$.

<u>Locally</u>, we can identify TM with $M\times\mathbb{R}^n$. In that sense, ~~TM can be identified with M×ℝⁿ~~. $0_{\gamma_{p,v}(t)}$ can all be considered as the $0\in\mathbb{R}^n$. Hence we have

$$dF(p,0)(v,0) = \frac{d}{dt}\Big|_{t=0} F(\gamma_{p,v}(t), 0)$$
$$= \frac{d}{dt}\Big|_{t=0} (\gamma_{p,v}(t), \gamma_{p,v}(t)) = (v, v).$$

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② For any $v \in T_pM$, Consider the curve in TM

$$(p,\ tv)$$

~~we have~~ whose intial tangent is $(0, v) \in T_{(p,0)}(TM)$.

Then
$$dF(p,0)\,(0,v) = \frac{d}{dt}\Big|_{t=0} F(p, tv)$$
$$= \frac{d}{dt}\Big|_{t=0} (p,\ \exp_p(tv))$$
$$= (0,\ v).$$

From the above ① and ②, ~~we have~~ if we consider $(v,0)$ & $(0,v)$ as column vectors, $dF(p,0)$ is the $2n \times 2n$ matrix such that

$$dF(p,0)\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} v \\ v \end{pmatrix}, \quad \forall v \in T_pM.$$

$$dF(p,0)\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \forall v \in T_pM.$$

This implies
$$dF(p,0) = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}.$$

Hence, in particular, $dF(p,0)$ is nonsingular.