

(II) Geodesics.

§1. Geodesic equations and Christoffel symbols.

Let $\gamma: (a, b) \rightarrow (M, g)$ be a regular smooth curve.
(i.e. $\dot{\gamma}(t) \neq 0, \forall t \in (a, b)$)

Recall its length is defined as

$$L(\gamma) := \text{Length}(\gamma) := \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\dot{\gamma}(t)}} dt.$$

In a local coordinate neighborhood

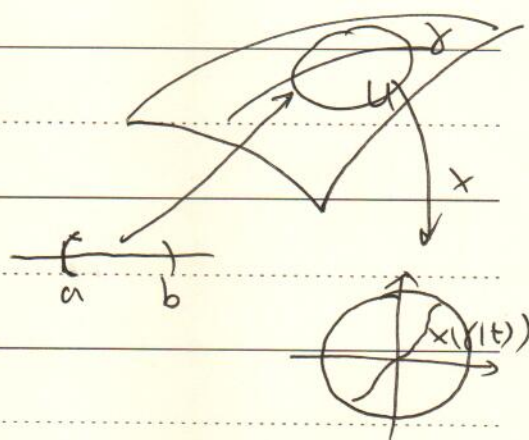
$$(U, x^1, \dots, x^n)$$

The curve can be written as

$$(x^1(\gamma(t)), \dots, x^n(\gamma(t))).$$

When $\gamma|_{(a,b)}$ is contained in U ,

we have $\dot{\gamma}(t) = \frac{dx^i(\gamma(t))}{dt} \frac{\partial}{\partial x^i}$

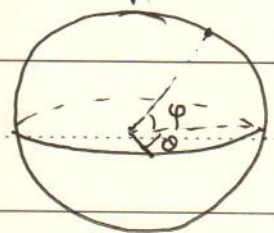


and
$$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t))) \dot{x}^i(t) \dot{x}^j(t)} dt.$$

Example: Consider the unit sphere $S^2 \subset \mathbb{R}^3$ with ~~spherical~~

Consider the coordinate neighborhood

$$\left\{ (\varphi, \theta) \mid \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \theta \in (0, 2\pi) \right\}$$



We have the induced Rie. metric

$$g = d\varphi \otimes d\varphi + \cos^2 \varphi d\theta \otimes d\theta$$

Consider a smooth curve $\gamma(t)$, $t \in (a, b)$ on S^n with the spherical coordinate $(\varphi(t), \theta(t))$. Then

$$L(\gamma) = \int_a^b \sqrt{\dot{\varphi}(t)^2 + \cos^2 \varphi(t) \dot{\theta}(t)^2} dt.$$

Observe that

$$L(\gamma) \geq \int_a^b |\dot{\gamma}(t)| dt \geq \left| \int_a^b \dot{\gamma}(t) dt \right| = |\gamma(b) - \gamma(a)|$$

where "=" holds iff $\underbrace{\dot{\theta}(t) = 0}$ & γ is monotonic.

$$\Leftrightarrow \theta(t) \equiv \text{const.}$$

Therefore, where $\gamma(a)$ and $\gamma(b)$ has the same coordinate θ , the shortest curve connecting them is the great circle passing through them. \square

A natural question is then; given two points $p, q \in M$,

- (i) does there exist a shortest curve @ connecting p, q ?
- (ii) if it exists, is it unique?

In order to find the shortest curve, we consider the critical point of the Length functional, $\text{Length}(\gamma)$. Note that $\text{Length}(\gamma)$ is ~~not so~~ a bit messy to handle with since it has a $\sqrt{\cdot}$ term as the integrand. In fact, we can consider the Energy functional instead:

$$\begin{aligned} E(\gamma) &:= \frac{1}{2} \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt \\ &= \frac{1}{2} \int_a^b g_{ij}(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t) dt. \end{aligned}$$

(In physics, $E(\gamma)$ is usually called "action of γ " where γ is considered as the orbit of a mass point. In physics, we have the so-called "least action principle").

In the following, we will explain why we can ~~not~~ consider

the critical value of $E(\gamma)$ instead of that of $L(\gamma)$.

Lemma 1.1. For each smooth curve $\gamma: (a, b) \rightarrow M$, we have

$$L(\gamma)^2 \leq 2(b-a)E(\gamma)$$

and " $=$ " holds if and only if $\sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} =: \|\dot{\gamma}(t)\| \equiv \text{const.}$

Proof. By Hölder's inequality,

$$\begin{aligned} L(\gamma) &= \int_a^b \|\dot{\gamma}(t)\| dt \leq \left(\int_a^b 1^2 dt \right)^{1/2} \left(\int_a^b \|\dot{\gamma}(t)\|^2 dt \right)^{1/2} \\ &= \sqrt{b-a} \sqrt{2E(\gamma)}. \end{aligned}$$

with equality ~~holds~~ precisely if $\|\dot{\gamma}(t)\| \equiv \text{const.}$ \square

Recall the length of a curve does not depend on the choice of the parametrizations. Hence, we only need to consider

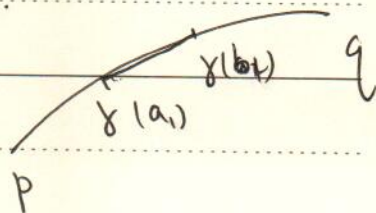
~~a~~ curves parametrized by arc length, i.e. $\|\dot{\gamma}(t)\| = 1$, in order to find shortest curves. In this case, we have

$$\left. \begin{aligned} b-a &= L(\gamma) \\ \text{and } L(\gamma)^2 &= 2(b-a)E(\gamma) \end{aligned} \right\} \Rightarrow L(\gamma) = 2E(\gamma).$$

Hence, ~~for~~ after parametrizing curves by arc length, it is enough to minimize $E(\gamma)$.

Moreover, observe that if $\gamma \in C_{p,q}$ is a shortest curve from p to q , $\gamma: (a, b) \rightarrow M$.

then for any $a \leq a_1 \leq b_1 \leq b$, γ is also a shortest curve from $\gamma(a_1)$ to $\gamma(b_1)$,



since, otherwise, we can shorten $\gamma|_{(a,b)}$ further.

So we can localize our problem, and ~~try to find~~ consider the case when $p, q \in U$.

Lemma 1.2. The Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0, \quad i=1, \dots, n$$

with

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} (g_{lk,j} + g_{jl,k} - g_{jk,l}),$$

and

$$g_{jlk} = \frac{\partial}{\partial x^k} g_{jl}.$$

Definition 1.1 (geodesics). A smooth curve $\gamma: [a,b] \rightarrow M$ ~~with~~ which satisfies (with $\dot{x}^i(t) = \frac{d}{dt} x^i(\gamma(t))$)

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0, \quad i=1, 2, \dots, n$$

is called a geodesic.

Remark: Christoffel is a German mathematician. He studied in Berlin, and worked in ETH Zürich, Strasbourg.

Riemann's 1854 Lecture was only published in 1868.

Christoffel published ~~an article~~ in Crelle's Journal

(Journal für die reine und angewandte Mathematik) in 1869

an article discussing the necessary condition Φ when two quadratic differential forms

$$F = \sum_{i,j} w_{ij} dx^i dx^j, \quad F' = \sum_{i,j} w'_{ij} dx^i dx^j$$

can be transformed into each other via independent ~~variable~~ changes. It was there he introduced the „Christoffel symbols“.

Influenced by Christoffel's work, Italian mathematician Gregorio Ricci - Curbasto published 6 articles during 1883-1888 on Christoffel's method, and introduced a new calculus. He ~~interp~~ interpreted Christoffel's algorithm into "covariant ~~derivative~~ differentiations". Ricci (1893) called it "absolute differential calculus".

Later in 1901, Ricci and his student Levi-Civita published Ricci's calculus in French in Klein's journal (Mathematische Annalen). It is now called "tensor analysis".

Einstein (1914) derives the geodesic equation ~~which~~ ~~via~~ using Christoffel symbols in his Berlin lecture ("§7. Geodesic line or equations of the path motion").

$$\frac{d^2 x_\mu}{ds^2} = \sum_{\nu} \left\{ \begin{matrix} \mu \nu \\ \tau \end{matrix} \right\} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}.$$

Levi-Civita (1916/17) realized the geometric meaning of Christoffel symbols: it determines the "parallel transport" of vectors. This pulls Christoffel and Ricci's discussion back to the track of geometry.

Proof of Lemma 2:

Let us first look at a general functional

$$I(x) = \int_a^b f(t, x(t), x'(t)) dt$$

where $x(t) := (x^1(t), x^2(t), \dots, x^n(t))$

Claim: the Euler-Lagrange equation of $I(x)$ is

$$\frac{d}{dt} \frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial x^i} = 0, \quad i = 1, \dots, n.$$

Proof of the claim: Consider $y(t) = (y^1(t), \dots, y^n(t))$ with

$$y(a) = y(b) = 0$$

Solving $\frac{d}{dz} \Big|_{z=0} I(x+zy) = 0,$

we have

$$\begin{aligned} 0 &= \int_a^b \left(\frac{\partial f}{\partial x^i} y^i(t) + \frac{\partial f}{\partial \dot{x}^i} \dot{y}^i(t) \right) dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x^i} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} \right) y^i(t) dt \end{aligned}$$

By the fundamental lemma of calculus of variations, we have

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0, \quad i=1, \dots, n.$$

This is the $E=L$ equation of I . \square

For our energy functional

$$E(x) = \int_a^b g_{ij}(x(t)) \dot{x}^j(t) \dot{x}^i(t) dt,$$

where $f(t, x(t), \dot{x}(t)) = g_{ij}(x(t)) \dot{x}^j(t) \dot{x}^i(t)$

we have

$$\frac{d}{dt} \left[g_{ik}(x(t)) \dot{x}^k(t) + g_{ji}(x(t)) \dot{x}^j(t) \right]$$

$$- g_{j,k,i}(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0$$

$$i=1, 2, \dots, n$$

Hence, $\underline{g_{ik,l} \ddot{x}^l \dot{x}^k + g_{ik} \ddot{x}^k} + g_{ji,l} \dot{x}^l \dot{x}^j + g_{ji} \ddot{x}^j$

$$- g_{j,k,i} \dot{x}^j \dot{x}^k = 0, \quad i=1, \dots, n.$$

\Rightarrow

$$\boxed{2g_{im} \ddot{x}^m + (g_{ik,j} + g_{ji,k} - g_{jk,i}) \dot{x}^j \dot{x}^k = 0}$$

$$l=1, \dots, n.$$

Multiply both sides by g^{il} and sum up over i , we have

$$\ddot{x}^l + \frac{1}{2} g^{il} (g_{ik,j} + g_{ji,k} - g_{jk,i}) \dot{x}^j \dot{x}^k = 0$$

$$l=1, \dots, n.$$

□

Remark: When we calculate the term $\Gamma_{jk}^i \dot{x}^j \dot{x}^k$, pay attention to the fact that

$$\Gamma_{jk}^i \dot{x}^j \dot{x}^k = \frac{1}{2} g^{il} (g_{j\ell,k} + g_{k\ell,j} - g_{\ell k,j}) \dot{x}^j \dot{x}^k$$

$$= \frac{1}{2} g^{il} (2 g_{j\ell,k} - g_{\ell k,j}) \dot{x}^j \dot{x}^k$$