Proof of $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$.

Denote by $h(t) = \gamma_v(\lambda t). = (y^1(t), \dots, y^n(t))$, where

$$y^i(t) = x^i(\lambda t).$$

We check $\ddot{y}^i(t) + \Gamma^i_{jk}(y(t))\, \dot{y}^j(t)\, \dot{y}^k(t)$

$$= \lambda^2 \ddot{x}^i(t) + \lambda^2 \Gamma^i_{jk}(x^i(\lambda t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0$$

That is h is also a geodesic. Moreover

$$h(0) = \gamma_v(0) = q, \quad \dot{h}(t) = \lambda v.$$

By the uniqueness of geodesics,

$$h(t) = \gamma_{\lambda v}(t).$$

Lemma (Homogeneity of a geodesic, [doC, Lemma 2.6 §3.2])
If the geodesic $\gamma(t, q, v)$ is defined on the interval $t \in (-\varepsilon, \varepsilon)$, then the geodesic $\gamma(t, q, \lambda v)$, $\lambda \in \mathbb{R}$, $\lambda > 0$, is defined on the interval $t \in (-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda})$ and

$$\gamma(t, q, \lambda v) = \gamma(\lambda t, q, v)$$