Hopf-Rinow

(i) $\Rightarrow$ (iv) metrically completeness $\Rightarrow$ geodesically completeness.

[PP', § 5.7.1]

Proof $\forall p$, Let $v \in T_pM$ be a unit vector. Suppose that $\exp_p tv$ can be defined on a maximal interval $[0, b)$. $\hookrightarrow$ This side is open due to the " local existence and uniqueness of geodesics " (ODE theory).

Suppose $b < \infty$. with $\exp_p t_0 v \in \Omega$,

Claim For any compact set $\Omega \subset M$, there exist $t_1 \in [t_0, b)$ s.t.

$$\exp_p t_1 v \notin \Omega.$$

(" every geodesic $c : [0, b) \to M$ defined on a maximal interval must leave every compact set if $b < \infty$ ").

This is true since otherwise, $\{\exp_p tv,\ t \in [t_0, b)\} \subset \Omega$.

Recall $\exists\, \delta_0 > 0$, s.t, $\forall\, q \in \Omega$, the Rie. polar coord. can be introduced on $B(q, \delta_0)$.

Therefore, the geodesic can be ~~$\exp_p tv|_{t\in[0,}$~~ extended further by length δ_0.

That is, $\forall\, t \in [t_0, b)$, we have $t + \delta_0 \in [t_0, b)$.

This contradicts to the assumption that $b < \infty$.

On the other hand, given $(t_n)_{n\in\mathbb{N}} \subset [0, b)$ converging to b,

$\{\gamma(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence since $d(\gamma(t_n), \gamma(t_m)) = |t_n - t_m|$

$\gamma(t_n) = \exp_p t_n v$

completeness $\Rightarrow \exists\, p_0 \in M$, s.t. $\gamma(t_n) \to p_0$. But $\gamma|_{t\in[0,b)}$ will always leave the compact set $B(p_0, \delta)$, which is a contradiction.