Riemannian Geometry Spring 2018

<u>Course information</u>: staff.ustc.edu.cn/~spliu/Teaching.html

Shiping Liu 刘世平 spliu@ustc.edu.cn Office: 1611

Course assessment: Homework + Mid-term Exam + Final Exam

<u>References</u>:

① Michael Spivak. A comprehensive introduction to differential geometry, volume I-V, third edition, Publish or Perish

② Jürgen Jost, Riemannian geometry and geometric analysis, Springer.

③ Manfredo Perdigão do Carmo, Riemannian geometry, Birkhäuser

④ Peter Petersen, Riemannian geometry, GTM171, Springer

⑤ 伍鸿熙,沈纯理,虞言林,黎曼几何初步.高等教育出版社.

<u>Tutorials</u>

Tutor: Yulin Gong 龚雨霖 gylustc@mail.ustc.edu.cn

Homeworks should be handed in every two weeks, after the lecture on Thursday. (First one on March 22, 2018)

First tutorial March 24, 2018

Introduction

On June 10, 1854, Georg Friedrich Bernhard Riemann delivered a lecture entitled

„Über die Hypothesen, welche der Geometrie zu Grund liegen"
(On the Hypotheses which lie at the Foundations of Geometry)
to the faculty of Göttingen University. This lecture was published later in 1866, and gives birth to Riemannian geometry.

This lecture was given by Riemannan as a probationary inaugural lecture for seeking the position of „Privatdocent". Privatdocent is a position in the German university system. It is a lecturer who received no salary, but is merely forwarded fees paid by those students who elected to attend his lectures. To attain such a position, one has to submit an inaugural paper (Habilitationsschrift) AND give a probationary inaugural lecture on a topic chosen by the faculty, from a list of 3 proposed by the candidate. The first 2 topics which Riemann submitted were ones on which he has already worked; The 3rd topic he chose was the fundations of geometry. Usually, the faculty chooses the first topic proposed by the candidate. However, contrary to all traditions, Gauss passed over the first two and chose instead the 3rd of Riemann's topics. So Riemann has to ~~ta~~ prepare a lecture on a topiz that he had not worked on before. In the end, Riemann finished his lecture in about seven more weeks.

Why did Gauss choose the 3rd topic? In fact, that is topic in which Gauss had been interested in for many years.

The single most important work in the history of differential geometry is Gauss' paper, in Latin, of 1827:

„Disquisitiones generales circa superficies curvas"

(General Investigations of Curved Surfaces)

The most influential result in Gauss' paper is the so-called „Theorema Egregium". Rougly speaking, this theorem asserts that the Gauss curvature of a surface is <u>determined</u> by its <u>first fundamental form</u>. This opens the door to „Intrinsic geometry" and provides possibility of studying more abstract spaces other than surfaces in $\mathbb{E}^3$. For example, one can study the geometry of a "flat torus", which is a topological torus associated with a "flat metric".

What Riemann did in his lecture is developing ~~a~~ higher-dim intrinsic geometry. Geometry presupposes the concept of <u>space</u>. In this course of Riemannian geometry, the space we study is a C^∞-manifold M (Hausdorff and second countable) ~~and~~ associated a Riemannian metric g. A vital step is to understand what is the extension of Gauss curvature in higher dimensional manifolds. The original definition of Gauss curvature using Gauss map is not ~~visible~~ available in higher dim. mflds. The expression of Gauss curvature in the Gauss equation is possibly extended to higher dim. spaces.

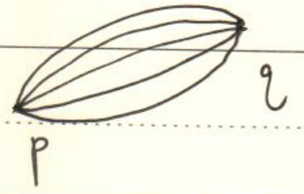
Actually, before discussing "curvature", we can already see a lot of information of the geometry of the underlying spaces from the Riemannian metric.

We will roughly follow the ~~s~~ scheme below :

(I) Riemannian metric.

From the Rie. metric, we can calculate the length of curves, and moreover, ~~vol~~ we obtain a natural volume measure.

(II) Geodesics



It's natural to look for shortest curves connecting two points. Geodesics are curves which are "locally" shortest. ~~I~~ They can be obtained from the First variation of the length functional. In order to explore the problem whether a geodesic is shortest, we will discuss exponential maps, normal coordinates, ~~etc~~. Hopf - Rinow theorem (1931) etc.

(III) Connections, Parallelism, and Covariant Derivatives.

We reinterpret the geodesic equations in terms of connections and covariant derivatives, or parallelism. In this process, we develop (Absolute) calculus on Rie. mflds.

(IV) Curvature.

When ~~we~~ we consider the Second Variation of the length functional. In this variational formula, a "curvature" term will appear, which is a generalization of Gaussian curvature of surfaces. We will discuss properties of Riemannian curvature tensor and various curvature notions

(V) Space forms and Jacobi fields

We will discuss the complete Rie. mflds with constant curvature, which are referred to as <u>space forms</u>. Those will be <u>model spaces</u> when we study the geometry of a general Rie. mfld. In this process, we discuss the theory of Jacobi fields – variational vector fields of ~~geo~~ family of geodesics.

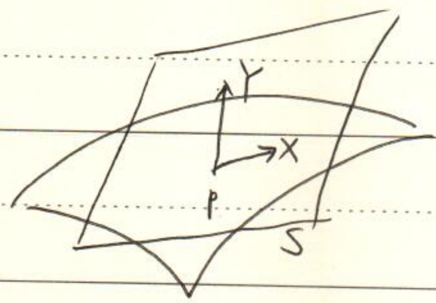
(VI) Comparison theorems.

We explore geometry of Rie. mflds with curvature bounds via comparing them with <u>space forms</u>.

(I) Riemannian Metric

§1 Definition

Recall in the theory of surfaces in $\mathbb{E}^3$, ~~once we assign~~



For a surface $S \subset \mathbb{E}^3$, $\forall p \in S$, and any two tangent vectors $X, Y \in T_pS$, we have the inner product

$$\langle X, Y \rangle_p \qquad (\langle \cdot, \cdot \rangle_p \text{ is the inner product of } \mathbb{E}^3)$$

This inner product $\langle \cdot, \cdot \rangle_p$ corresponds to the first fundamental forms of S at p. Based on this inner product, one can compute the lengths of a curve in S, the area of a domain in S etc.

Now, let us consider a C^∞-manifold M^n (n dim) (Hausdorff, second countable).

<u>Definition 1.1</u> (Riemannian Metric) A Riemannian metric g

on M is a "C^∞ assignment": For each tangent vector space

T_pM, $p \in M$

of M, we assign an inner product $g_p(\cdot,\cdot) = \langle \cdot, \cdot \rangle_p$, which is smoothly dependent on p in the following sense: $f(p) := \langle X_p, Y_p \rangle_p = g_p(X_p, Y_p)$ is a smooth function on $U \subset M$ for any smooth tangent vector fields X, Y on $U \subset M$. □

Remark 1.1. Recall that by inner product, we mean $g_p(\cdot,\cdot)$ is symmetric, positive definite, and bilinear.

What it looks like in local coordinates: Given $p \in M$. For any coordinate neighborhood $U \ni p$, let its coordinate functions be $\{x^1, \dots, x^n\}$. Then the tangent vector space T_pM is spanned by $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$

Its dual space – the cotangent vector space – T_p^*M is spanned by $\{dx^1, \dots, dx^n\}$.

Then we denote

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_p = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(p) := g_{ij}(p).$$

and for any smooth tangent vector fields

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}$$

we have

$$\langle X_p, Y_p \rangle_p = X^i(p)\, Y^j(p) \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_p$$

$$= g_{ij}(p)\, X^i(p)\, Y^j(p)$$

[Here we use the Einstein summation convention: An index occuring twice in a product is to be summed from 1 up to the space dimension.

Therefore, in local ~~coo~~ coordinates, we can consider the Rie. metric g as

$$g = g_{ij}\, dx^i \otimes dx^j ,$$

where (i) " $g_p(\cdot,\cdot)$ depends smoothly on p" is equivalent to say

" $g_{ij}(p)$, $\forall i,j$ is smooth on $U \ni p$ ".

(ii) " $g_p(\cdot,\cdot)$ is an inner product " is equivalent to say

- $g_{ij} = g_{ji}$, i.e. the matrix $[\,g_{ij}(p)\,]$ is symmetric at any $p \in U$.
- The matrix $(g_{ij}(p))$ is positive definite at any $p \in U$.

Hence, we can reformulate the definition of a Rie. metric as

Definition′ (Riemannian Metric) A Riemannian metric g on a C^∞-mfld M is a smooth $(0,2)$-tensor satisfying

$$g(X,Y) = g(Y,X), \quad g(X,X) \geq 0 \ \& \ g_p(X,X) = 0 \Leftrightarrow X(p) = 0.$$

for any smooth tangent vector fields X, Y.

we will always assume C^∞

Definition 1.2 A Riemannian manifold is a differentiable manifold equipped with a Riemannian metric.

Remark: a couple (M, g).

§2. Examples

(1) $M = \mathbb{R}^n$, $\forall p \in \mathbb{R}^n$, $T_p\mathbb{R}^n = \mathbb{R}^n$. The standard inner

product on $\mathbb{R}^n$ gives a standard Riemannian metric g_0

$$g_0(X,Y) = \sum_i X^i Y^i = X^T Y.$$

Another way to see it: $\mathbb{R}^n$ is covered by a single coordinate $(x^1, \cdots, x^n)$.

Matrix: $(g_{ij}) = (\delta_{ij}) = Id_{n\times n}$

Tensor: $g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$

More generally, (g_{ij}) can be any positive definite, symmetric, $n\times n$ matrix $A = (a_{ij})$

$$g = a_{ij}\, dx^i \otimes dx^j$$

$$g(X,Y) = X^T A Y.$$

(2) Induced Metric. Let $f: M^n \to N^{n+k}$ be a smooth immersion (i.e. $df_p : T_pM^n \to T_{f(p)}N$ is injective for any $p \in M$)

Let (N, g_N) be a Riemannian mfld (e.g. $(N, g_N) = (\mathbb{R}^{n+k}, g_0)$)

We can define the pull-back metric $f^* g_N$ on M as below

$$(f^* g_N)_p (X_p, Y_p) = (g_N)_{f(p)} \left(df_p(X_p), df_p(Y_p) \right)$$

One can verify that $f^* g_N$ is a Riemannian metric on M.

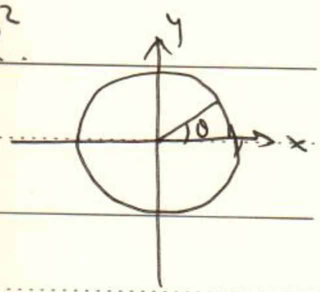
$$\left((f^* g_N)_p (X_p, X_p) = 0 \Leftrightarrow df_p(X_p) = 0 \underset{df_p \text{ is injective}}{\Leftrightarrow} X_p = 0 \right)$$

Definition 2.1. We call $f^* g_N$ an induced metric on M with respect to the smooth immersion $f: M^n \to N^{n+k}$.

A special case. $M \subset N$ is an immersed submanifold. Then the inclusion map $i: M \to N$ is an immersion. ~~And~~

In this case,
the induced metric $(i^* g)_p$ is just the restriction of $(g_N)_p$ on $T_pM \subset T_pN$.

Example (i) Let $M = S^1$ be the unit circle in $\mathbb{R}^2$.
Choose a coordinate neighborhood, $0 < \theta < 2\pi$.
Then the inclusion map is given by



$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$

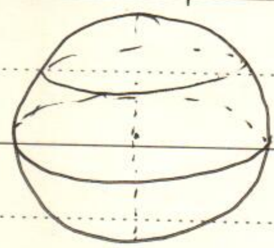
Then we have $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

and $$g_{S^1} = (dx \otimes dx + dy \otimes dy)\big|_{S^1}$$

$$= d\theta \otimes d\theta.$$

(ii) Let $M = S^2$ be the unit sphere in $\mathbb{R}^3$.
Choose a coordinate neighborhood $\{(\theta, z) \mid 0 < \theta < 2\pi, -1 < z < 1\}$

$$\begin{cases} x = \sqrt{1-z^2}\,\cos\theta \\ y = \sqrt{1-z^2}\,\sin\theta \\ z = z \end{cases}$$



Then the induced metric g on S^2 is

$$g_{S^2} = \frac{1}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta$$

(Exercise).

(3) Product metric.

Let (M, g_M), (N, g_N) be two Rie. mflds. Let $M \times N$ be the Cartesian product.
Let $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$ be the natural projections.

Then $M \times N$ has the following product metric g:

$$g_{(p,q)}(X,Y) = (g_M)_p\left(d\pi_1(X), d\pi_1(Y)\right) + (g_N)_q\left(d\pi_2(X), d\pi_2(Y)\right)$$

$\forall (p,q) \in M \times N, \quad \forall X, Y \in T_{(p,q)}(M \times N)$.

For example, the torus $T^n = S^1 \times \cdots \times S^1$ has a product metric based on the induced metric on S^1. Such ~~torus~~ tori are flat tori.

(4) If g_1, g_2 are two Rie. metrics of M, So does $ag_1 + bg_2$, $a, b > 0$.

§3 When are two Riemannian manifolds "equivalent"?

Definition 3.1 (Isometry) Let (M, g_M), (N, g_N) be two Riemannian manifolds. Let $\varphi : M \to N$ be a diffeomorphism (i.e. φ is bijective, C^∞, and φ^{-1} is also C^∞).

If $\varphi^* g_N = g_M$ (that is, $(g_M)_p(X,Y) = (g_N)_{\varphi(p)}(d\varphi_p(X), d\varphi_p(Y))$ $\forall p \in M, \forall X, Y \in T_pM$)

then we call φ an isometry.

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~~Local isometry. Given $p \in M$~~

§4 Existence of Riemannian Metrics

Theorem 4.1. A C^∞-manifold M (Hausdorff, second countable) has a Riemannian metric.

Proof:

Let $\{U_\alpha\}$ be a locally finite covering of M by coordinate neighborhoods. That is, any point p of M has a neighborhood U such that $U \cap U_\alpha \neq \phi$ at most for a finite number of indices.

Let $\{\phi_\alpha\}$ be a C^∞ partition of unity on M subordinate to the covering $\{U_\alpha\}$. That is.

(i) $\phi_\alpha \geq 0$, $\phi_\alpha = 0$ on $M \setminus \overline{U_\alpha}$

(ii) $\sum_\alpha \phi_\alpha(p) = 1$, $\forall p \in M$.

On each U_α, we can define a Rie. metric $g_\alpha(\cdot,\cdot)$ induced by the local coordinates (e.g. $(U_\alpha, x_\alpha^i) \to g_\alpha = \sum_i dx_\alpha^i \otimes dx_\alpha^i$).
Let us set

$$g_p(X,Y) := \sum_\alpha \phi_\alpha(p)\,(g_\alpha)_p(X,Y), \quad \forall p \in M, \quad \forall X, Y \in T_pM. \qquad (*)$$

It is direct to verify that this construction defines a Rie. metric on M. (In fact, the main point is to check that g is positive definite. This is because the summation in $(*)$ is actually a finite sum, and $\sum_\alpha \phi_\alpha = 1 \Rightarrow \exists \beta$ s.t. $\phi_\beta(p) > 0$. Hence $g_p \geq \phi_\beta(p)\, g_p > 0$.

$\square$

Remark. Whitney (1936) showed that any n-dimensional C^∞-mfld M^n can be embedded into $\mathbb{R}^{2n+1}$. Thus M^n always has a Rie metric induced from the standard Rie metric g_0 of $\mathbb{R}^{2n+1}$.

On the other hand, given a Riemannian manifold (M^n, g_M), the Rie. metric g_M is usually different from the metric induced from g_0 of $\mathbb{R}^{2n+1}$. In fact, Nash's embedding theorem tells that for any Rie. mfld (M^n, g_M), there exists a N, s.t. (M^n, g_M) can be isometrically embedded into $(\mathbb{R}^N, g_0)$. In other words, there exists an embedding $\varphi: M^n \to \mathbb{R}^N$ s.t. $g_M = \varphi^* g_0$.

Nevertheless, the intrinsic point of view in the above proof offers great conceptual and ~~techniqu~~ technical advantages over the approach of submanifold geometry of Euclidean spaces.

§5. The metric structure.

The Riemannian metric g on M induces a natural distance function d. That is a function $d: M\times M \to \mathbb{R}$ satisfying for any $p, q, r \in M$

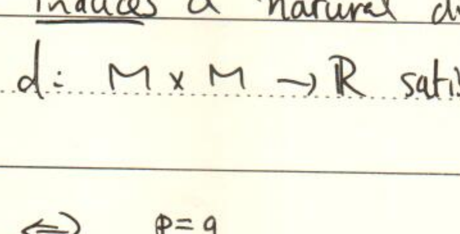
(i) $d(p,q) \geq 0$, and $d(p,q)=0 \Leftrightarrow p=q$

(ii) $d(p,q) = d(q,p)$

(iii) $d(p,q) \leq d(p,r) + d(r,q)$

To show this fact, Let's ~~fo~~ consider the length of curves in M.
Let $\gamma: [a,b] \to M$ be a smooth (parametrized) curve in M.
For any $t\in[a,b]$, we have the tangent vector

$$\dot{\gamma}(t) := d\gamma\left(\frac{d}{dt}\right) \in T_{\gamma(t)}M$$



We always assume the parametrization is regular, i.e. $\dot{\gamma}(t)\neq 0, \forall t$.
Then the length of γ is

$$\mathrm{Length}(\gamma) := \int_a^b |\dot{\gamma}(t)|\, dt$$

$$= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_{\gamma(t)}}\, dt.$$

Lemma 5.1: The quantity $\mathrm{Length}(\gamma)$ does not depend on the choice of parametrization.

Proof: Let ~~$\gamma(t)$~~ $\gamma_1 : [c,d] \to M$ $(c\leq d)$ is another regular parametrization of the same curve. Then there exists a smooth function $t_1 = t_1(t) : [a,b] \to [c,d]$ s.t.

$$\gamma_1(t_1(t)) = \gamma(t)$$

Since both parametrization of are regular, we have

$$\dot{\gamma}_1(t_1) \neq 0, \quad \dot{\gamma}(t) \neq 0$$

Observe that

$$\boxed{\dot{\gamma}(t) = \dot{\gamma}_1(t_1) \cdot \frac{dt_1}{dt}} \qquad (*)$$

(By definition, $(*)$ can be checked as below: for any smooth function f, $\dot{\gamma}(t) \cdot f = d\gamma\left(\frac{d}{dt}\right) \cdot f = \frac{d}{dt} f(\gamma(t)) = \frac{d}{dt} f(\gamma_1(t_1(t)))$

$$= \frac{d}{dt_1} f(\gamma_1(t_1)) \frac{dt_1}{dt} = d\gamma_1\left(\frac{d}{dt_1}\right) \cdot f \cdot \frac{dt_1}{dt}$$

$$\Rightarrow \dot{\gamma}(t) = \dot{\gamma}_1(t_1) \frac{dt_1}{dt} \Big).$$

Hence, we have $\frac{dt_1}{dt} \neq 0$, which means either $\frac{dt_1}{dt} > 0$ or $\frac{dt_1}{dt} < 0$. As we assume $a \leq b$, $c \leq d$, we have here $\frac{dt_1}{dt} > 0$.

Now we calculate

$$\text{Length}(\gamma_1) = \int_c^d \sqrt{\langle \dot{\gamma}_1(t_1), \dot{\gamma}_1(t_1) \rangle_{\gamma_1(t_1)}} \, dt_1$$

$$= \int_a^b \sqrt{\left\langle \frac{dt}{dt_1} \dot{\gamma}(t), \frac{dt}{dt_1} \dot{\gamma}(t) \right\rangle_{\gamma(t)}} \, \frac{dt_1}{dt} \, dt$$

$$= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} \left| \frac{dt}{dt_1} \right| \left| \frac{dt_1}{dt} \right| dt$$

$$= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} \, dt = \text{Length}(\gamma). \qquad \square$$

Exercise: Let $\varphi : (M, g_M) \to (N, g_N)$ be an isometry, γ be a smooth curve in M. Show that $\text{Length}_M(\gamma) = \text{Length}_N(\varphi(\gamma))$.

<u>Arclength parametrization</u>. Now we look for a "standard" parametrization for a given curve.

Consider smooth curve $\gamma : [a,b] \to M$.

We can define the arclength function of γ:

$$s(t) := \int_a^t \sqrt{\langle \dot\gamma(\tau), \dot\gamma(\tau)\rangle_{\gamma(\tau)}}\, d\tau$$

Then $s = s(t) : [a,b] \to [0, \mathrm{Length}(\gamma)]$ is a strictly increasing function. Denote by $t = t(s)$ its inverse function. Then we can reparametrize $\gamma(t)$ as

$$\gamma_1(s) = \gamma(t(s)), \quad s \in [0, \mathrm{Length}(\gamma)]$$

<u>Proposition 5.1</u> $\langle \dot\gamma_1(s), \dot\gamma_1(s)\rangle_{\dot\gamma_1(s)} \equiv 1$

<u>Proof</u>. We calculate

$$\langle \dot\gamma_1(s), \dot\gamma_1(s)\rangle_{\gamma_1(s)} = \langle \dot\gamma(t)\frac{dt}{ds}, \dot\gamma(t)\frac{dt}{ds}\rangle_{\gamma(t(s))}$$

$$= \left(\frac{dt}{ds}\right)^2 \cdot \underbrace{\langle \dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}}_{=\left(\frac{ds}{dt}\right)^2}$$

$$= 1.$$

□

<u>Remark</u>: The length of a (continuous and) piecewise smooth curve is defined as the <u>sum</u> of the lengths of the smooth pieces.

On a Rie. mfld (M,g), the <u>distance</u> between two points p, q can be defined:

$$d(p,q) := \inf\{\mathrm{Length}(\gamma) \mid \gamma \in C_{p,q}\}$$

where $C_{p,q} := \{\gamma : [a,b] \to M \mid \gamma$ piecewise smooth curve with $\gamma(a) = p,\ \gamma(b) = q\}$.

It can be checked immediately that $d: M\times M \to \mathbb{R}$ satisfies

(i) $d(p,p)=0$, $d(p,q)\geq 0$

(ii) $d(p,q)=d(q,p)$

(iii) $d(p,q)\leq d(p,r)+d(r,q)$ (by definition)

To show that d is indeed a distance function, it remains to prove:

$$d(p,q)=0 \Rightarrow p=q$$

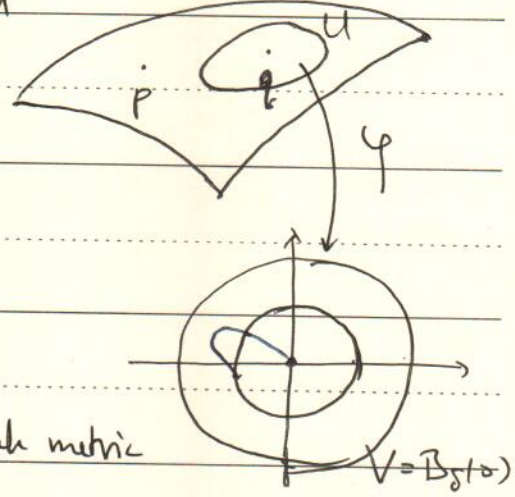
or equivalently, $p\neq q \Rightarrow d(p,q)>0$.

<u>Theorem 5.1</u>. (M,d) is a metric space.

<u>Proof</u>. It remains to show $p\neq q \Rightarrow d(p,q)>0$

There exists a coordinate neighborhood U of q, with coordinate map φ such that

$$\begin{cases} \varphi(q)=0 \ni B_\delta(0) =: V\subset \mathbb{R}^n \\ p\notin U \end{cases}$$



Denote $\varphi^{-1}: B_\delta(0)\to U$.

Then on $B_\delta(0)$, we have the pull-back metric

$$(\varphi^{-1})^* g := h.$$

Let $\gamma:[a,b]\to B_\delta(0)$ be a curve connecting $0=\varphi(q)$ and a point on $\partial B_{\frac{\delta}{2}}(0)$. ~~Let $\gamma|_{[a,c]}$ be~~ Let c be the ~~first~~ smallest number with $\gamma(c)\in\partial B_{\frac{\delta}{2}}(0)$. Then we have

$$\mathrm{Length}_h(\gamma)\geq \mathrm{Length}_h(\gamma|_{[a,c]})=\int_a^c \sqrt{\langle\dot\gamma,\dot\gamma\rangle_h}\,dt$$

Observe that there exists a positive constant ε s.t.

$$\langle \dot\gamma, \dot\gamma\rangle_h \geq \varepsilon \langle \dot\gamma, \dot\gamma\rangle_{g_0} \quad \text{on } B_\delta(0).$$

(Exercise: Let K be an open set of $\mathbb{R}^n$, $\{g_{ij}(x)\}$ be n^2 continuous functions on K, such that, the matrix $[g_{ij}(x)]$ is symmetric for any $x \in K$.

(1) ~~If~~ Denote by $\lambda(x)$, $\Lambda(x)$ the smallest, largest eigenvalues of $[g_{ij}(x)]$. Show that both λ and Λ are continuous functions on K

(2). Suppose K is compact and $[g_{ij}(x)]$ is positive definite for any $x \in K$. Show that: there exist positive constants ε_1, ε_2 such that

$$\varepsilon_1 |v|^2 \leq \sum_{i,j} g_{ij}(x) v^i v^j \leq \varepsilon_2 |v|^2$$

for any $x \in K$ and any $(v^1, \cdots, v^n) = v \in \mathbb{R}^n$.)

Therefore, we have

$$\text{Length}_h(\gamma) \geq \sqrt{\varepsilon} \cdot \int_a^c \langle \dot\gamma, \dot\gamma\rangle_{g_0} dt = \sqrt{\varepsilon}\, \text{Length}_{g_0}(\gamma|_{[a,c]})$$

$$\geq \sqrt{\varepsilon} \cdot \frac{\delta}{2}.$$

Any curve connecting $p, q \in M$ must intersect with $\varphi^{-1}(\partial B_{\frac{\delta}{2}}(0))$ at some point. Hence, we have

$$d(p,q) \geq \frac{\delta\sqrt{\varepsilon}}{2} > 0. \qquad \square$$

Moreover, we have the following property.

Proposition 5.2 Let (M,g) be a Rie. mfld. For given $p \in M$, the function
$$f(\cdot) := d(\cdot, p) : M \to \mathbb{R}$$
is continuous.

Proof: We need to show $f(q_i) \to f(q)$ while q_i tends to q

(in the sense of the manifold topology).

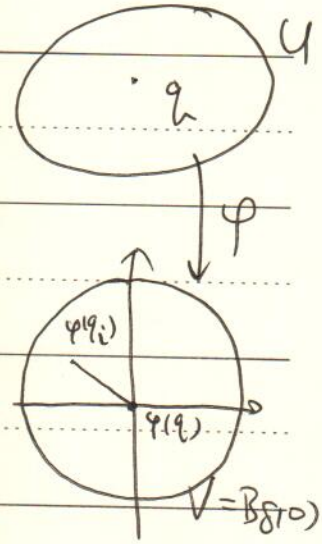
By triangle inequality, $|f(q_i)-f(q)| \le d(q_i,q)$.

So it suffice to prove $d(q_i,q)\to 0$ as $i\to\infty$.

Pick a coordinate neighborhood U of q s.t.

$\exists\ \varphi: U \longrightarrow B_\delta(0) =: V \subset \mathbb{R}^n$

$\varphi(q)=0$.



W. l. o. g., we assume

$$\varphi(q_i) \in B_{\frac{1}{i}}(0).$$

Denote by $h := (\varphi^{-1})^* g$.

Choose $\tilde\gamma_i : [0,1] \to V$ be the curve

$$\tilde\gamma_i(t) = t\,\varphi(q_i).$$

(Note $\tilde\gamma_i(0)=\varphi(q)$, $\tilde\gamma_i(1)=\varphi(q_i)$).

Then ~~we have~~ $\exists\ \varepsilon_2>0$ s.t.

$$\text{Length}_h(\tilde\gamma_i) = \int_0^1 \sqrt{\langle \dot{\tilde\gamma}_i, \dot{\tilde\gamma}_i\rangle_h}\, dt \le \sqrt{\varepsilon_2} \int_0^1 \sqrt{\langle \dot{\tilde\gamma}_i, \dot{\tilde\gamma}_i\rangle_{g_0}}\, dt$$

$$\le \sqrt{\varepsilon_2}\,\frac{1}{i}$$

Therefore, we have

$$d(q_i,q) \le \text{Length}_h(\tilde\gamma_i) \le \frac{\sqrt{\varepsilon_2}}{i} \to \infty \quad \text{as } i\to\infty. \quad \square$$

<u>Corollary 5.1</u> The topology on M induced by the distance function d coincides with the original manifold topology of M.

<u>Proof</u>. The continuity of $f(\cdot)=d(\cdot,p)$ tells every open set of the topology induced by d is again open of the manifold topology. By the proof of <u>Prop. 5.2</u>, one can see the other way around: every open set of the manifold topology is open in the topology induced by d.

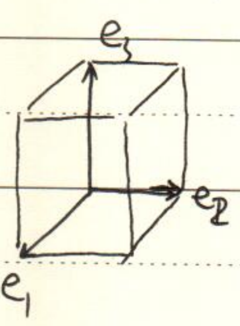
$\square$

Remark: It actually suffice to show that in each coordinate neighborhood, the topology induced by d coincides with the one in $\mathbb{R}^n$, which is induced by the Euclidean distance. We know for any point in this coordinate, $\exists$ positive constants $\varepsilon_1, \varepsilon_2$ with

$$\varepsilon_1 |v|^2 \leq g_{ij} v^i v^j \leq \varepsilon_2 |v|^2, \quad \forall\, v \in \mathbb{R}^n.$$

By our previous argument, this implies the Rie. distance d and the Euclidean distance control each other. Hence the two topology coincides.

§6. Riemannian Measure, Volume

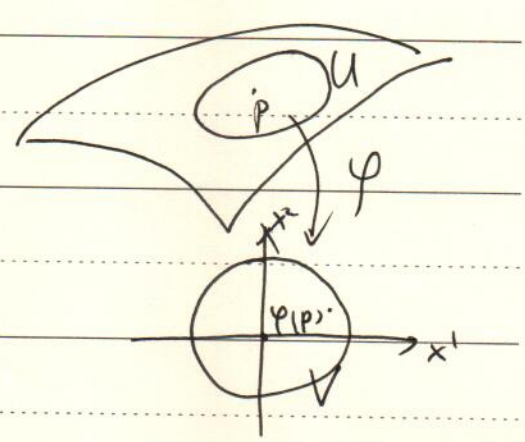


For any $p \in M$, (T_pM, g) is an ~~inner~~ vector space with inner products. Consider an orthonormal basis

$$\{e_1, \ldots, e_n\}$$

of (T_pM, g). The volume of the parallelepiped spanned by $e_1, \cdots, e_n$, $\mathrm{vol}(e_1, \cdots, e_n) = 1$.

Now we hope to develop a natural notion of integration on a Riemannian manifold.



Locally, such an integration should be the integration over an Euclidean subset V:

$$\int_V \ldots \, dx^1 \cdots dx^n.$$

So we need to know the length of the tangent vectors $\left\{\frac{\partial}{\partial x^i}(p),\ i = 1, \ldots, n\right\}$

and the volume of the paraflelepiped spanned by them.

We can write

$$\frac{\partial}{\partial x^i}(p) = a_i^j e_j$$

Then $g_{ik}(p) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle (p) = \langle a_i^j e_j, a_k^l e_l \rangle (p)$

$$= a_i^j a_k^l \delta_{jl} = \sum_l a_i^l a_k^l$$

Matrix form:

$$[g_{ik}] = AA^T, \quad \text{with } A = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^n \end{pmatrix}$$

Therefore, we have

$$\mathrm{vol}\left(\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^n}(p)\right) = \det(a_i^j)(p)\, \underbrace{\mathrm{vol}(e_1, \ldots, e_n)}_{=1}$$

$$= \sqrt{\det(g_{ij})(p)}.$$

Volume of "small" compact domain.

$\forall p \in M$, Let $(U, x^1, \ldots, x^n)$ be a coordinate neighborhood. $x: U \to \mathbb{R}^n$.

Consider cpt set $K \subset U$ such that $x(K)$ is measurable (in $\mathbb{R}^n$).

The we define its volume as

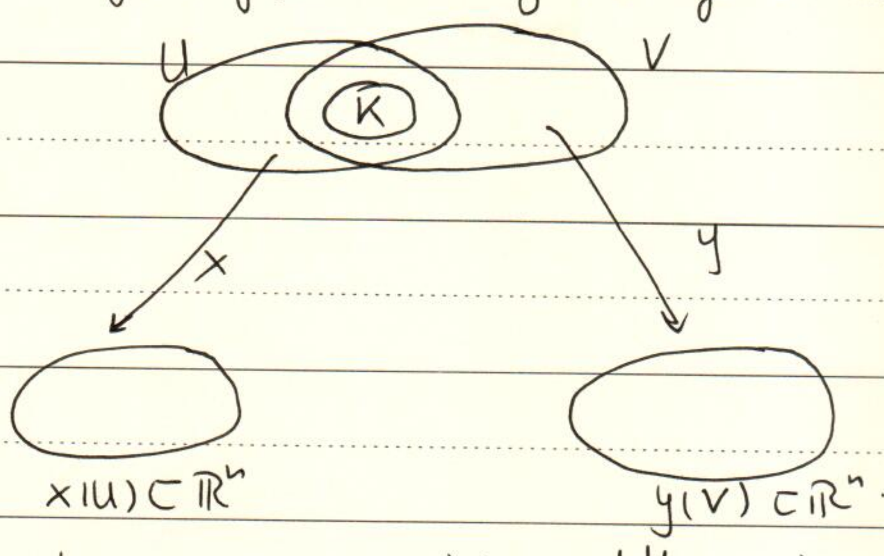
$$(*) \quad \mathrm{vol}(K) := \int_{x(K)} \sqrt{\det(g_{ij}) \circ x^{-1}}\, \underbrace{dx^1 \cdots dx^n}_{\text{Lebesgue measure on } \mathbb{R}^n}.$$

Well-definedness?

Proposition 6.1. The definition $(*)$ does not depends on the choice of the coordinate.

Proof: ~~Let us~~ Suppose we have another coordinate neighborhood

$(V, y^1, \cdots, y^n)$ containing K. $y : V \to \mathbb{R}^n$.



$y \circ x^{-1} : x(U) \longrightarrow y(V)$ is a diffeomorphism.

Observe that $\dfrac{\partial}{\partial x^i}(p) = \dfrac{\partial (y^k \circ x^{-1})}{\partial x^i}(x(p)) \dfrac{\partial}{\partial y^k}(p)$

We have

$$g^x_{ij}(p) := \left\langle \frac{\partial}{\partial x^i}(p), \frac{\partial}{\partial x^j}(p) \right\rangle$$

$$= \left\langle \frac{\partial}{\partial y^k}(p), \frac{\partial}{\partial y^l}(p) \right\rangle \frac{\partial (y^k \circ x^{-1})}{\partial x^i}(x(p)) \frac{\partial (y^l \circ x^{-1})}{\partial x^j}(x(p))$$

$$= \frac{\partial (y^k \circ x^{-1})}{\partial x^i}(x(p)) \cdot \frac{\partial (y^l \circ x^{-1})}{\partial x^j}(x(p))\, g^y_{kl}(p).$$

Denote the Jacobian matrix of the map $y \circ x^{-1}$ by

$$J(x(p)) = \begin{pmatrix} \frac{\partial (y^1 \circ x^{-1})}{\partial x^1} & \frac{\partial (y^1 \circ x^{-1})}{\partial x^2} & \cdots & \frac{\partial (y^1 \circ x^{-1})}{\partial x^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial (y^n \circ x^{-1})}{\partial x^1} & \frac{\partial (y^n \circ x^{-1})}{\partial x^2} & \cdots & \frac{\partial (y^n \circ x^{-1})}{\partial x^n} \end{pmatrix}$$

We obtain $\left[g^x_{ij}(p) \right] = J(x(p))^T \left[g^y_{kl}(p) \right] J(x(p))$

Hence $\sqrt{\det (g^x_{ij})(p)} = \left| \det J(x(p)) \right| \cdot \sqrt{\det (g^y_{kl}(p))}$

Therefore we have

$$\int_{y(K)} \sqrt{\det(g_{ij}^{y})\circ y^{-1}}\, dy^1 \cdots dy^n \overset{y = y\circ x^{-1}}{=} \int_{x(K)} \sqrt{\det(g_{ij}^{y})\circ y^{-1}(y\circ x^{-1})} \cdot \left|\det J_{x(p)}\right| dx^1 \cdots dx^n$$

$$= \int_{x(K)} \sqrt{\det(g_{ij}^{x})\circ x^{-1}}\, dx^1 \cdots dx^n .$$

$\square$

Volume of "larger" compact domain.

Now let us consider the case when K can not be contained in a single coordinate neighborhood. Let

$$\{U_\alpha, x_\alpha^1, \ldots, x_\alpha^n\}_\alpha$$

be a locally finite covering of M by coordinate neighborhoods. Let $\{\phi_\alpha\}_\alpha$ be a C^∞ partition of unity on M subordinate to the covering $\{U_\alpha\}$.

Then we define

$$\mathrm{vol}(K) := \sum_\alpha \int_{x_\alpha(K\cap U_\alpha)} \phi_\alpha \circ x_\alpha^{-1} \sqrt{\det(g_{ij}^{x_\alpha})\cdot x_\alpha^{-1}}\, dx_\alpha^1 \cdots dx_\alpha^n .$$

Proposition 6.2. This definition does not depend on the choice of the covering of coordinate neighborhoods and partition of unity.

Proof. Let $\{V_\beta, y_\beta^1, \ldots, y_\beta^n\}_\beta$ be another locally finite covering of M by coordinate neighborhood, and $\{\psi_\beta\}_\beta$ be a partition of unity of M subordinate to $\{V_\beta\}$

Then, we compute

$$\sum_\beta \int_{y_\beta(K\cap V_\beta)} \psi_\beta \circ y_\beta^{-1} \sqrt{\det(g_{ij}^{y_\beta}) \circ y_\beta^{-1}} \, dy_\beta^1 \cdots dy_\beta^n$$

$$= \sum_\beta \int_{y_\beta(K\cap V_\beta)} \sum_\alpha \phi_\alpha \circ y_\beta^{-1} \left(\psi_\beta \sqrt{\det(g_{ij}^{y_\beta})}\right) \circ y_\beta^{-1} \, dy_\beta^1 \cdots dy_\beta^n$$

We can Exchange the order of the two summations, since each is a finite sum.

$$= \sum_\alpha \int_{y_\beta(K\cap V_\beta)} \sum_\beta \psi_\beta \circ y_\beta^{-1} \, \phi_\alpha \circ y_\beta^{-1} \sqrt{\det(g_{ij}^{y_\beta})} \circ y_\beta^{-1} \, dy_\beta^1 \cdots dy_\beta^n$$

$$\overset{\text{Change of variables}}{=} \sum_\alpha \int_{x_\alpha(K\cap U_\alpha)} \sum_\beta \psi_\beta \circ x_\alpha^{-1} \cdot \phi_\alpha \circ x_\alpha^{-1} \sqrt{\det(g_{ij}^{x_\alpha})} \circ x_\alpha^{-1} \, dx_\alpha^1 \cdots dx_\alpha^n$$

$$= \sum_\alpha \int_{x_\alpha(K\cap U_\alpha)} \sqrt{\det(g_{ij}^{x_\alpha})} \circ x_\alpha^{-1} \, dx_\alpha^1 \cdots dx_\alpha^n .$$

$\square$

Let us ~~deno~~ denote by $C_0^0(M)$ the vector space of continuous functions on M with compact support.

For any $f \in C_0^0(M)$, we define

$$\int_M f := \sum_\alpha \int_{x^\alpha(U_\alpha)} (\phi_\alpha f) \circ x_\alpha^{-1} \sqrt{\det g_{ij}^{x_\alpha} \circ x_\alpha^{-1}} \, dx_\alpha^1 \cdots dx_\alpha^n .$$

From the above discussions, we know this is well-defined.

Moreover, since $\phi_\alpha \geq 0$, we know

$$f \geq 0 \implies \int_M f \geq 0.$$

Therefore, we obtain a positive linear functional

$$\mathcal{I} : C_0^0(M) \to \mathbb{R}$$

$$\mathcal{I}(f) := \int_M f$$

By ~~Riez~~ Riesz representation theorem, there exists a unique Borel measure dvol, s.t.

$$J(f) = \int_M f = \int_M f \, dvol$$

for any $f \in C_0^0(M)$.

Remark: In each coordinate neighborhood, the integration ~~over~~ w.r.t. dvol can be considered as the integration w.r.t. the n-form

$$\Omega_0 = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n .$$

Notice that when we change the coordinate, we have

$$dy^1 \wedge \cdots dy^n = \det(J(x(p)) \cdot dx^1 \wedge \cdots \wedge dx^n$$

Jacobian of the map $y \circ x^{-1}$.

That is, Ω_0 may change sign when we change from one coordinate to the other one.

Particularly, we can have a globally defined n-form Ω_0 when M is orientable. In this case,

$$\int_M f \, dvol = \int_M f \, \Omega_0 .$$

(Recall: On an orientable mfld M, let $\{e_1, \ldots, e_n\}$ be an orthonormal frame fields, and $\{\omega^1, \cdots, \omega^n\}$ be its dual. Then

$$\Omega_0 = \omega^1 \wedge \cdots \wedge \omega^n .$$
). □

Once ~~the~~ the measure dvol is obtained, the machine of measure theory is intiated. We define the L^p norm $1 \le p < \infty$ of $f \in C_0^0(M)$ as

$$\|f\|_{L^p} := \left(\int_M |f|^p \, dvol \right)^{1/p} .$$

We can take the completion of $C_0^\infty(M)$ w.r.t L^p-norm, the resulting space is called $L^p(M)$.

- In particular for $p=2$, we can define an inner product:

$$\langle f_1, f_2 \rangle_{L^2} := \int_M f_1 \cdot f_2 \, dvol, \quad \forall f_1, f_2 \in L^2(M)$$

This verifies $L^2(M)$ to be a Hilbert Space.

§7. Divergence Theorem

Let X be a smooth tangent vector fields of M. ~~Let~~ The divergence of X is defined as a C^∞ function $\operatorname{div} X$ on M as below: Let $(U, x^1, \cdots, x^n)$ be a coordinate neighborhood, we have the volume element

$$\Omega_0 = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n.$$

~~Definition~~ The divergence $\operatorname{div} X : M \to \mathbb{R}$ is a C^∞ fct on M such that

$$(\operatorname{div} X)\Omega_0 = d\,(i(X)\Omega_0),$$

where $i(X)$ is the interior product w.r.t. X. (i.e., the contraction of a differential form with the vector field X).

That is, for any vector fields $Y_1, \cdots, Y_{n-1}$, we have

$$i(X)\Omega_0(Y_1, \cdots, Y_{n-1}) := \Omega_0(X, Y_1, \cdots, Y_{n-1}).$$

Remarks: (1) When we change coordinates, Ω_0 may change sign, but this does not matter for the definition of $\operatorname{div} X$. The global definition of $\operatorname{div} X$ does not require ~~that~~ the orientability of M.

(2) Let us consider the expression of $\operatorname{div} X$ in local coordinate.
Let $X = X^i \frac{\partial}{\partial x^i}$.

$$i(X)\Omega_0 = i\left(X^i \frac{\partial}{\partial x^i}\right) \sqrt{\det(g_{kl})}\, dx^1 \wedge \cdots \wedge dx^n$$

<u>Lemma</u>. we have

$$i\left(\frac{\partial}{\partial x^i}\right)(dx^1 \wedge \cdots \wedge dx^n) = (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

<u>Proof</u>: Let $Y_1, \cdots, Y_{n-1}$ be any $(n-1)$ smooth vector fields.
we compute $i(\partial_i)\, dx^1 \wedge \cdots \wedge dx^n (Y_1, \cdots, Y_{n-1})$

$$= dx^1 \wedge \cdots \wedge dx^n (\partial_i, Y_1, \cdots, Y_{n-1})$$

$$= \sum_{\sigma \in S(n)} (\operatorname{sgn} \sigma)\, dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(n)} (\partial_i, Y_1, \cdots, Y_{n-1})$$

$$= \sum_{\substack{\sigma \in S(n) \\ \sigma(1) = i}} \operatorname{sgn}(\sigma)\, dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)} (Y_1, \cdots, Y_{n-1})$$

$$= \sum_{\tau \in S(n-1)} (-1)^{i-1} \operatorname{sgn}(\tau)\, dx^{\tau(1)} \otimes \overbrace{\cdots \otimes dx^{\tau(i)} \otimes \cdots}^{} \otimes dx^{\tau(n)} (Y_1, \cdots, Y_{n-1})$$

$$= (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n (Y_1, \cdots, Y_{n-1}) \quad \square$$

Hence $i(X)\Omega_0 = \sum_i X^i \sqrt{\det(g_{kl})}\, (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$.

~~On the other hand~~ Furthermore, we obtain

$$d(i(X)\Omega_0) = \sum_i (-1)^{i-1} \sum_k \frac{\partial}{\partial x^k}(\sqrt{G}\, X^i)\, dx^k \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \cdots \wedge dx^n$$

$$\sqrt{G} := \sqrt{\det(g_{kl})}$$

$$= \sum_i \frac{\partial}{\partial x^i}(\sqrt{G}\, X^i)\, dx^1 \wedge \cdots \wedge dx^n.$$

$$=: (\operatorname{div} X)\, \Omega_0$$

Hence $\operatorname{div} X = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G}\, X^i)$.

Notice that in particular, when $M=\mathbb{R}^n$, $(g_{ij}) = (\delta_{ij})$, we have for $X = X^i \frac{\partial}{\partial x^i}$,

$$\operatorname{div} X = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} X^i = \sum_{i=1}^{n} \frac{\partial X^i}{\partial x^i}$$

This reduces to the classical divergence.

(3) By Cartan's magical formula

$$\underset{\substack{\uparrow \\ \text{Lie derivative of diff. forms.}}}{\mathcal{L}_X} \omega = i(X)\, d\omega + d\, i(X)\omega .$$

we have $\mathcal{L}_X \Omega_0 = i(X) \underbrace{d\Omega_0}_{=0} + d\, i(X)\omega = \operatorname{div}(X)\Omega_0$.

This tells us that the divergence of a vector field, is the "infinitesimal" changing rate of the volume element along the vector field.

<u>Theorem 7.1</u> (Divergence Theorem) Let X be a smooth vector fields on (M,g). Then

$$\int_M \operatorname{div}(X)\, d\mathrm{vol} = 0.$$

<u>Proof</u>. Let $\{U_\alpha\}$ be a locally finite covering of M by coordinate neighborhood, $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then we have

$$X = \sum_\alpha \phi_\alpha X.$$

Since X has cpt support and the summation above is finite, we have

$$\int_M \operatorname{div}\left(\sum_\alpha \phi_\alpha X\right) d\mathrm{vol} = \sum_\alpha \int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, d\mathrm{vol}.$$

So it is enough to show ~~for~~ $\int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, d\mathrm{vol}$ holds for each α.

W.l.o.g., we assume the support of X is contained in a coordinate neighborhood $(U, x^1, \dots, x^n)$, and $X = X^i \frac{\partial}{\partial x^i}$.
By definition, we have

$$\int_M \mathrm{div}(X)\, d\mathrm{vol} = \int_U \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(X^i \sqrt{G})\, d\mathrm{vol}$$

$$= \int_{x(U)} \left(\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(X^i \sqrt{G}) \sqrt{G}\right)\circ x^{-1} dx^1 \cdots dx^n$$

$$= \int_{x(U)} \frac{\partial}{\partial x^i}(X^i \sqrt{G} \circ x^{-1})\, dx^1 \cdots dx^n$$

$$= 0. \qquad \square$$

Gradient ~~vectorfs~~ fields of a function.

Let $f \in C^\infty(M)$. The gradient vector field of f, ~~and~~ $\mathrm{grad} f$, is ~~a sm~~ defined as a smooth vector field such that

$$\langle \mathrm{grad} f, Y \rangle \left(= g(\mathrm{grad} f, Y)\right) = Y(f).$$

Expressions in local coordinate: $(U, x^1, \dots, x^n)$, $Y = Y^j \frac{\partial}{\partial x^j}$

Suppose $\mathrm{grad} f = X^i \frac{\partial}{\partial x^i}$.

Then by definition

$$\langle \mathrm{grad} f, Y \rangle = g_{ij} X^i Y^j = Y(f) = Y^k \frac{\partial f}{\partial x^k}$$

That is $$(g_{ij} X^i) Y^j = \frac{\partial f}{\partial x^k} Y^k, \quad \forall Y.$$

$$\Rightarrow \quad g_{ij} X^i = \frac{\partial f}{\partial x^j}.$$

Recall $[g_{ij}]$ is a positive definite matrix. Denote by $[g^{ij}]$ its inverse matrix, i.e. $g^{ik} g_{kj} = \delta^i_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$

Next, we compute

$$g_{ij}X^i = \frac{\partial f}{\partial x^j} \Rightarrow \underbrace{g^{kj} g_{ij}}_{=\delta^k_i} X^i = g^{kj}\frac{\partial f}{\partial x^j}$$

$$\delta^k_i X^i = X^k.$$

$$\Rightarrow X^k = g^{kj}\frac{\partial f}{\partial x^j}.$$

Hence $\boxed{\operatorname{grad} f = g^{kj}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^k}}$.

Remark (1) For the case $M = \mathbb{R}^n$, $(g_{ij}) = (\delta_{ij})$, we have $\operatorname{grad} f = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i} = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)$.

(2) "The gradient ~~vertical~~ vector field is vertical to ~~its~~ the level set of a function."

Proposition 7.1. Let $f \in C^\infty(M)$, c be a regular value of f. Then the vector field $\operatorname{grad} f$ is vertical to the level set $f^{-1}(c)$.

Proof. Since c is a regular value of f, $f^{-1}(c)$ is a submanifold of M. Let $X \in Tf^{-1}(c) \subset TM$. We know

$$Xf = 0 \quad \text{on } f^{-1}(c).$$

Therefore, $\langle \operatorname{grad} f, X\rangle = X(f) = 0$. on $f^{-1}(c)$. □

(3) We say a one form η is dual to a vector field X if

$$\langle X, Y\rangle = \eta(Y), \text{ for any vector field } Y.$$

In particular, we see

$$\cancel{\langle X, Y\rangle} \langle \operatorname{grad} f, Y\rangle = Y(f) = df(Y), \ \forall Y.$$

That is, df is dual to $\operatorname{grad} f$.

Corollary 7.1 Let $f \in C_0^\infty(M)$. Then

$$\int_M \mathrm{div}\,(\mathrm{grad} f)\, d\mathrm{vol} = 0.$$

Definition 7.1. The Laplacian of a smooth function f is

$$\Delta f := \mathrm{div}\,(\mathrm{grad} f)$$

Remark (1) In local coordinate $(U, x^1, \ldots, x^n)$, we have

$$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left((\mathrm{grad} f)^i \sqrt{G} \right)$$

$$= \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(g^{ij} \frac{\partial f}{\partial x^j} \sqrt{G} \right)$$

$$= \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij} \frac{\partial f}{\partial x^j} \right).$$

In particular, for the case $M = \mathbb{R}^n$, $g_{ij} = \delta_{ij}$, we have

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial x^{i\,2}}$$

(2). $\int_M \Delta f \, d\mathrm{vol} = 0, \quad \forall f \in C_0^\infty(M)$

(3). For any smooth fct f and vector field X, we have

$$\mathrm{div}(fX) = f \,\mathrm{div} X + \langle \mathrm{grad} f, X \rangle.$$

Proof. $\mathrm{div}(fX) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(f X^i \sqrt{G} \right)$

$$= \frac{1}{\sqrt{G}} \frac{\partial f}{\partial x^i} X^i \sqrt{G} + f \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(X^i \sqrt{G} \right)$$

$$= X(f) + f \,\mathrm{div} X$$

$$= \langle \mathrm{grad} f, X \rangle + f \,\mathrm{div} X. \qquad \square$$

Theorem 7.2 (Green's formula) Let f, h be two smooth functions, at least one of which has compact support. Then

$$\int_M f \Delta h \, d\mathrm{vol} = - \int_M \langle \mathrm{grad} f, \mathrm{grad} h \rangle \, d\mathrm{vol}$$

$$= \int_M (\Delta f) \cdot h \, d\mathrm{vol}.$$

$$\int_M f \Delta h \, d\mathrm{vol}_M = - \int_M g \langle \mathrm{grad} f, \mathrm{grad} h \rangle \, d\mathrm{vol}_M + \int g(\mathrm{grad}\, h, \nu) f.$$

Proof: Applying $\mathrm{div}(fX) = f\,\mathrm{div}X + \langle \mathrm{grad} f, X\rangle$, to $X = \mathrm{grad}\,h$, we have

$$\begin{aligned}\mathrm{div}(f\,\mathrm{grad}\,h) &= f\,\mathrm{div}(\mathrm{grad}\,h) + \langle \mathrm{grad} f, \mathrm{grad}\,h\rangle \\ &= f\Delta h + \langle \mathrm{grad} f, \mathrm{grad}\,h\rangle.\end{aligned}$$

Since $f\,\mathrm{grad}\,h$ has cpt support, we can apply divergence theorem to derive the Green's formula. □

Remark: Δ is called the Laplace - Beltrami operator.
On a compact mfld (M,g), we have

$$\langle \Delta f, h\rangle_{L^2} = \langle f, \Delta h\rangle_{L^2}, \quad \forall f, h \in C^\infty(M).$$

(i.e. Δ is self-adjoint)

$$\langle \Delta f, f\rangle_{L^2} = -\langle \mathrm{grad} f, \mathrm{grad} f\rangle_{g} = \underbrace{-\int_M |\mathrm{grad} f|^2\, d\mathrm{vol}}_{\leq 0}$$

(i.e. $-\Delta$ is positive).

Remark: Divergence theorem and Green's formula can be extended to compact Rie. mflds with boundary.

Theorem 7.3: Let M be a compact Rie. mflds with C^∞ boundary ∂M. Let ν be the outward normal vector field on ∂M, X be a smooth vector field on M. Then

$$\int_M (\mathrm{div}X)\, d\mathrm{vol}_M = \int_{\partial M} g(X, \nu)\, d\mathrm{vol}_{\partial M}.$$

Remark: Let (M^n, g) be a cpt Rie. submfld of (N, g_N). Then ∂M has a Rie. metric induced from g_N. Therefore we have a natural $d\mathrm{vol}_{\partial M}$. As a corollary, we have

$$\int_M f\Delta h\, d\mathrm{vol}_M = -\int_M g\langle \mathrm{grad} f, \mathrm{grad}\,h\rangle\, d\mathrm{vol}_M + \int_{\partial M} g(\mathrm{grad}\,h, \nu) f\, d\mathrm{vol}_{\partial M}.$$

Last Lecture: More explanation on the calculation (p. 24)

$$\sum_{\substack{\sigma \in S(n) \\ \sigma(1)=i}} \operatorname{sgn}(\sigma)\, dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)} (Y_1, \cdots, Y_{n-1})$$

Notice that $\{\sigma(2), \cdots, \sigma(n)\} = \{1, 2, \cdots, \hat{i}, \cdots, n\}$

~~So $\sigma(2), \dots, \sigma(n)$ can be considered as a permutation of~~

So $\sigma(2), \cdots, \sigma(n)$ is produced by a permutation τ of $\{1, 2, \cdots, \hat{i}, \cdots, n\}$

Moreover $\operatorname{sgn}(\sigma) = (-1)^{i-1} \operatorname{sgn} \tau$

Hence $\displaystyle\sum_{\substack{\sigma \in S(n) \\ \sigma(1)=i}} \operatorname{sgn}(\sigma)\, dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)} (Y_1, \cdots, Y_{n-1})$

$$= \sum_{\tau \in S(n-1)} (-1)^{i-1} \operatorname{sgn}\tau\, dx^{\tau(1)} \otimes \cdots \otimes dx^{\tau(i-1)} \otimes dx^{\tau(i+1)} \otimes \cdots \otimes dx^{\tau(n)} (Y_1, \cdots, Y_{n-1})$$

$$= (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n (Y_1, \cdots, Y_{n-1}).$$