RIEMANNIAN GEOMETRY
EXCERCISE 1

1. (i) An immersion $f : N \to \mathbb{R}^{n+1}$ of an n dimensional smooth manifold into $\mathbb{R}^{n+1}$ is called a hypersurface. Suppose f can be expressed locally in a coordinate neighborhood $(U, u^1, \dots, u^n)$ as

$$x^k = f^k(u^1, \dots, u^n), \;\; 1 \le k \le n+1,$$

where $(x^1, \dots, x^{n+1})$ are the coordinates in $\mathbb{R}^{n+1}$. Let g_0 be the standard Eulidean metric on $\mathbb{R}^{n+1}$. Show

$$(f^* g_0)_{|U} = \sum_{k,i,j} \frac{\partial f^k}{\partial u^i} \frac{\partial f^k}{\partial u^j} du^i \otimes du^j.$$

(ii) Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Consider the coordinate neighborhood $(U, y^1, \dots, y^n)$ where

$$U := \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : x^{n+1} \neq 1\},$$

and

$$(y^1, \dots, y^n) := \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}} \right).$$

Prove the induced metric on S^n in $(U, y^1, \dots, y^n)$ is

$$\frac{4}{(1 + |y|^2)^2} \sum_{i=1}^{n} dy^i \otimes dy^i.$$

2. (i) On $\mathbb{R}^2 \setminus \{\text{half line}\}$ we have polar coordinates (r, θ). In these coordinates, compute the Riemannian metric on it induced from the Euclidean metric on $\mathbb{R}^2$.

(ii) A *surface of revolution* consists of a profile curve

$$c(t) = (r(t), 0, z(t)) : I \to \mathbb{R}^3,$$

where $I \subset \mathbb{R}$ is open and $r(t) > 0$ for all t. By rotating this curve around the z-axis, we get a surface that can be represented as

$$(t, \theta) \to f(t, \theta) = (r(t) \cos\theta, r(t) \sin\theta, z(t)).$$

Suppose the curve $c(t)$ is parametrized by arc length. Compute the Riemannain metric of this surface, in the above coordinate, from the Euclidean metric on $\mathbb{R}^2$.
(iii) The unit sphere $S^2 \subset \mathbb{R}^3$ can be though of as a surface of revolution by revolving

$$t \to R \left(\sin\left(\frac{t}{R}\right), 0, \cos\left(\frac{t}{R}\right) \right).$$

Use (ii) to derive the induced metric.

(iv)Consider the surface of revolution produced by the profile curve

$$t \to R \left(\sinh\left(\frac{t}{R}\right), 0, \cosh\left(\frac{t}{R}\right) \right).$$

Compute the metric induced from $(\mathbb{R}^3, dx \otimes dx + dy \otimes dy - dz \otimes dz)$.

3. Let $(U, x = (x^1, \ldots, x^n))$ be a chart of a Riemannian manifold (M, g). Let $E_1, \ldots, E_n$ be an *orthonormal frame* on U, that is, a collection of vector fields defined on the common domain U such that they form an orthonormal basis for the tangent spaces T_pM for all $p \in U$. Let Ω_0 be the volume n-form defined on U. Show that

$$\Omega_0(X_1, \ldots, X_n) = \det\left[g(X_i, E_j)\right],$$

holds for any n vector fields $X_1, \ldots, X_n$ on U.