RIEMANNIAN GEOMETRY
EXCERCISE 2

1. (Christoffel symbols) Let $(U, x = (x^1, \dots, x^n))$ be a chart of a Riemannian manifold M. Let

$$(x^1, \dots, x^n) \to (y^1, \dots, y^n) \tag{0.1}$$

be a smooth coordinate change, and the Riemannian metric can be written as

$$g_{ij}(x)dx^i \otimes dx^j \text{ and } h_{\alpha\beta}(y)dy^\alpha \otimes dy^\beta$$

respectively.

(i) Show the transformation formula of g^{ij} under the coordinate change (0.1) is

$$g^{ij}(x) = h^{\alpha\beta}(y(x))\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}.$$

(ii) Compute the transformation formulae of the Christoffel symbols Γ^i_{jk} under the coordinate change (0.1). Do they define a tensor?

(iii) Let $\gamma : [a, b] \to U$ be a smooth curve. Denote $\dot{x}^i(t) := \frac{d}{dt}x^i(\gamma(t))$. Compute the transformation formula of

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)$$

under the coordinate change (0.1).

2. (Lobatchevski plane) Let M be the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, with the Riemannian metric

$$\frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

(i) Compute the Christoffel symbols and write down the system of differential equations satisfied by the geodesics.

(ii) Prove that the curves $\gamma : (0, +\infty) \to M,\ t \mapsto (x_0, e^{ct})$, where $x_0 \in \mathbb{R}$, $c > 0$, are geodesics.

(iii) Let $0 < a < b < +\infty$. Let γ be as in (ii) and $\sigma : [a, b] \to M$ be a piecewise smooth curve connecting $\gamma(a)$ and $\gamma(b)$. Prove that

$$\text{Length}(\sigma_{|[a,b]}) \geq \text{Length}(\gamma_{|[a,b]}).$$

Characterize the cases when the equality holds.

(iv) At $p = (0, 1) \in M$, compute $\exp_p((0, \eta))$, where $(0, \eta) \in T_pM$ (we identify T_pM with $\mathbb{R}^2$ here); Recall the map $d\exp_p((0, \eta)) : T_{(0,\eta)}(T_pM) \to T_pM$. For $(0, 1) \in T_{(0,\eta)}(T_pM)$, compute $d\exp_p((0, \eta))((0, 1))$.

(v) Prove that the transformation

$$z \to \frac{az + b}{cz + d},\ z = x + iy, ad - bc = 1,$$

is an isometry of M.

(vi) Use (ii) and (v) to deduce the that the upper unit semicircle are geodesics.

3. (Geodesics on $P^n(\mathbb{R})$)

(i) Prove that the antipodal mapping $A : S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n.

(ii) Introduce a Riemannian metric on the real projective space $P^n(\mathbb{R})$ such that the natural projection $\pi : S^n \to P^n(\mathbb{R})$ is a local isometry.
(iii) Show that the geodesics of $P^n(\mathbb{R})$ are periodic with period π.

4. Assume that (M, g) has the property that all normal geodesics exist for a fixed time $\epsilon > 0$. Show that (M, g) is geodesically complete.

5. Let (M, g) be a metrically complete Riemannian manifold and $\tilde{g}$ is another metric on M such that $\tilde{g} \geq g$. Show that $(M, \tilde{g})$ is also metrically complete.

6. Let (M, g) be a Riemannian manifold which admits a proper Lipschitz function $f : M \to \mathbb{R}$. Show that (M, g) is complete. (Recall that a function between topological spaces is called proper if inverse images of compact subsets are compact.)