RIEMANNIAN GEOMETRY
EXCERCISE 3

1. Let M be a smooth manifold with an affine connection ∇. Let $X, Y \in \Gamma(TM)$. Let $U \subset M$ be an open subset. Prove that if $Y_{|U} = 0$, then $(\nabla_X Y)_{|U} = 0$.

2. (Covariant derivatives of tensor fields via parallel transport) Recall that for an isomorphism $\varphi : V \to W$ between two vector spaces V and W, there is an adjoint isomorphism

$$\varphi^* : W^* \to V^*,$$

between their dual spaces. For $\alpha \in W^*$, we have

$$\varphi(\alpha)(v) := \alpha(\varphi(v)), \ \ \forall \ v \in V.$$

Then, for any $v_i \in V$, $\alpha^j \in V^*$, we define

$$\widetilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s) = \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s).$$

By linearity, we can extend $\widetilde{\varphi}$ to be defined on all (r, s)-tensor, $\otimes^{r,s} V$, over V. This defines an isomorphism

$$\widetilde{\varphi} : \otimes^{r,s} V \to \otimes^{r,s} W.$$

Let M be a smooth manifold with an affine connection ∇. Let $c : I \to M$ be a smooth curve in M with $c(0) = p \in M$ and $\dot{c}(0) = X_p \in T_p M$. Recall that the parallel transport

$$P_{c,t} : T_{c(0)} M \to T_{c(t)} M,$$

is an isomorphism. As described above, we can extend it to be an isomorphism

$$\widetilde{P}_{c,t} : \otimes^{r,s} T_{c(0)} M \to \otimes^{r,s} T_{c(t)} M.$$

For any $A \in \Gamma(\otimes^{r,s} TM)$, we define

$$\nabla_{X_p} A := \lim_{h \to 0} \frac{1}{h} \left(\widetilde{P}_{c,h}^{-1} A(c(h)) - A(p) \right).$$

Let $Y \in \Gamma(TM)$, $w, \eta \in \Gamma(T^*M)$. Consider the $(1, 2)$-tensor filed $K := Y \otimes w \otimes \eta$.

(i) Show that

$$\nabla_{X_p} K = \nabla_{X_p} Y \otimes w \otimes \eta + Y \otimes \nabla_{X_p} w \otimes \eta + Y \otimes w \otimes \nabla_{X_p} \eta.$$

(ii) Let $C : \Gamma(\otimes^{1,2} TM) \to \Gamma(\otimes^{0,1} TM)$ be the contraction map that pairs the first vector with the first covector. For example, $CK = w(Y)\eta$. Show that

$$\nabla_{X_p}(CK) = C(\nabla_{X_p} K).$$

3. (Induced connection) Let M, N be two smooth manifold and $\varphi : N \to M$ be a smooth map. A vector field along φ is an assignment

$$x \in N \mapsto T_{\varphi(x)} M.$$

Let $\{E_i\}_{i=1}^n$ be a frame field in a neighborhood U of $\varphi(x) \in M$. Then for any $y \in \varphi^{-1}(U)$, we have

$$V(x) = V^i(x) E_i(\varphi(x)).$$

Let $u \in T_x N$. We define

$$\widetilde{\nabla}_u V := u(V^i)(x) E_i(\varphi(x)) + V^i(x) \nabla_{d\varphi(u)} E_i, \tag{0.1}$$

where ∇ is an affine connection on M.

(i) Check that $\widetilde{\nabla}_u V$ is well defined, i.e., (0.1) is independent of the choices of $\{E_i\}$.

(ii) Let g be a Riemannian metric on M. Prove that if ∇ on M is compatible with g, then for vector fields V, W along φ, and $u \in T_x N$, we have
$$u\langle V, W\rangle = \langle \widetilde{\nabla}_u V, W\rangle + \langle V, \widetilde{\nabla}_u W\rangle.$$

(iii) Prove that if ∇ on M is torsion free, then for any $X, Y \in \Gamma(TN$, we have
$$\widetilde{\nabla}_X d\varphi(Y) - \widetilde{\nabla}_Y d\varphi(X) - d\varphi\left([X, y]\right) = 0.$$

4. Let S^n be the sphere with the induced metric g from the Euclidean metric in $\mathbb{R}^{n+1}$. We denote by $\overline{\nabla}$ the canonical Levi-Civita connection on $\mathbb{R}^{n+1}$. For any $X, Y \in \Gamma(T\mathbb{S}^n)$, one can extend X, Y to smooth vector field $\overline{X}, \overline{Y}$ on $\mathbb{R}^{n+1}$, at least near $\mathbb{S}^n$.

By locality, the vector $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any $p \in \mathbb{S}^n$ depends only on $\overline{X}(p) = X(p)$ and the vectors $\overline{Y}(q) = Y(q)$ for $q \in \mathbb{S}^n$. That is, $\overline{\nabla}_{\overline{X}}\overline{Y}$ is independent of the extension of X, Y we choose. So we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\overline{X}}\overline{Y}$ at points on $\mathbb{S}^n$.

We define $\nabla_X Y$ to be the orthogonal projection of $\overline{\nabla}_X Y$ onto the tangent space of $\mathbb{S}^n$, i.e.,
$$\nabla_X Y := \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathbf{n}\rangle \mathbf{n},$$
where $\mathbf{n}$ is the unit out normal vector on $\mathbb{S}^n$.

(i) Prove that ∇ is an affine connection on $\mathbb{S}^n$.

(ii) Prove that ∇ is the Levi-Civita connection of $(\mathbb{S}^n, g)$.