§4. Existence of geodesics in given homotopy class

In complete Rie. manifolds, any two points p, q can be connected by a shortest geodesic. In this part, we discuss refinement of this result: That is, we discuss the existence of such geodesics in given homotopy class.

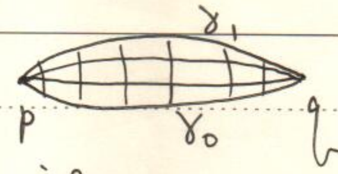
Definition 4.1 Two curves γ_0, γ_1 on a manifold M with common initial and end points p and q, i.e. two continuous maps

$$\gamma_0, \gamma_1 : I = [0,1] \longrightarrow M.$$

with $\gamma_0(0) = \gamma_1(0) = p$, $\gamma_1(1) = \gamma_1(1) = q$, are called homotopic if there exists a continuous map

$$\Gamma : I \times I \longrightarrow M.$$

with
$$\Gamma(0,s) = p, \quad \Gamma(1,s) = q \quad \forall s \in I$$
$$\Gamma(t,0) = \gamma_0(t), \quad \Gamma(t,1) = \gamma_1(t) \quad \forall t \in I.$$



Two closed curves c_0, c_1 in M, i.e. two continuous maps

$$c_0, c_1 : S^1 \longrightarrow M.$$

are called homotopic if there exists a continuous map

$$c : S^1 \times I \rightarrow M$$

with $c(t,0) = c_0(t)$, $c(t,1) = c_1(t)$ for all $t \in S^1$.

(S^1, as usual, is the unit circle parametrized by $[0, 2\pi)$) □

Remark: The concept of homotopy defines an equivalence relation on the set of all curves in M with fixed initial and end points as well as on the set of all closed curves in M.

With the example of torus in mind, let's first consider the existence of closed geodesic on a compact Riemannian manifold.

Theorem 4.1. Let M be a compact Rie. manifold, Then every homotopy class of closed curves in M contains a curve which is a shortest curve in its homotopy class and a geodesic.

(Later used in Synge Theorem, see Lemma 2 on page 146 in Chap(IV))

As a preparation, we first show

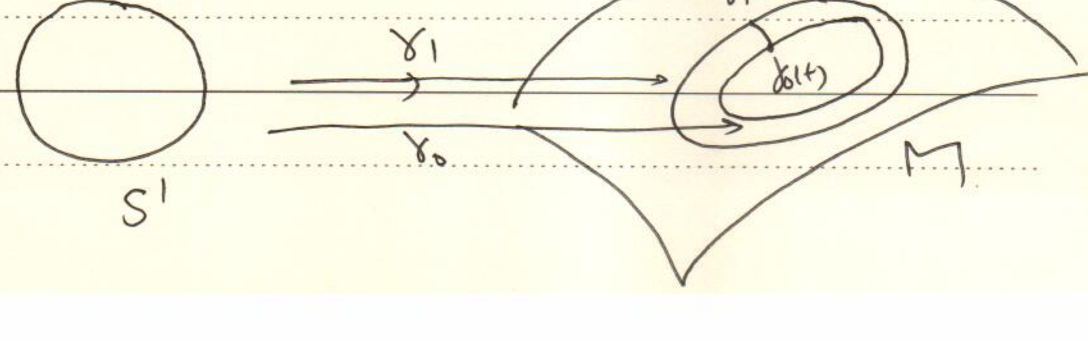
Lemma 4.1. Let M be a compact Rie. mfld. Let $\rho_0 > 0$ be the constant with the following property: any two points $p, q \in M$ with $d(p,q) \leq \rho_0$ can be connected by precisely one geodesic of shortest path. Let $\gamma_0, \gamma_1 : S^1 \to M$ be closed curves with

$$d(\gamma_0(t), \gamma_1(t)) \leq \rho_0 \quad \forall\ t \in S^1$$

Then γ_0 and γ_1 are homotopic.

Remark: The existence of such ρ_0 on a compact Rie. mfld has been proven in Corollary 3 on page 48. In fact, we know moreover, this geodesic depends continuously on (p, q).

Proof.

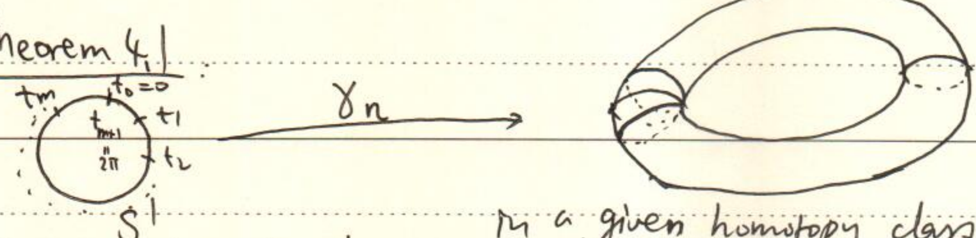


$\forall\, t \in S^1$, let $c_t(s) : I \longrightarrow M$ be the unique shortest ~~geodesic from~~ curve (which is, therefore, a geodesic) from $\gamma_0(t)$ to $\gamma_1(t)$, as usual parametrized proportionally to arc length. Recall that c_t depends continuously on its endpoints, hence on t.

$$\Gamma(t,s) := c_t(s)$$

is continuous and yields the desired homotopy. $\square$

Proof of Theorem 4.1



Consider the lengths of the curves in a given homotopy class: they are numbers in $[0, +\infty)$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a minimizing sequence for arc lengths in this homotopy class. All $(\gamma_n)_{n\in\mathbb{N}}$ are parametrized proportionally to arc length.

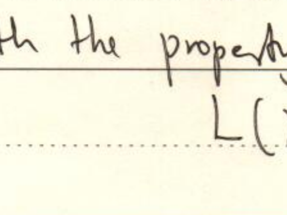
We may assume each γ_n is piecewise geodesic: for each γ_n, we may find

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 2\pi$$

with the property that

$$L(\gamma_n|_{[t_{j-1},t_j]}) \le \rho_0/2, \quad j = 1, \dots, m+1.$$

Replacing $\gamma_n|_{[t_{j-1},t_j]}$ by the shortest geodesic arc between $\gamma_n(t_{j-1})$ and $\gamma_n(t_j)$, we obtain a new closed curve $\tilde\gamma_n$.



Δ-ineq. $\Rightarrow$ $d(\gamma_n(t), \tilde\gamma_n(t)) \le \rho_0$

Lemma 4.1 $\Rightarrow$ $\tilde\gamma_n$ is homotopic to γ_n and the length of $\tilde\gamma_n$ is no longer than γ_n.

We may thus assume that for any γ_n, $\exists$ points $p_{0,n}, \cdots, p_{m,n}$ for which $d(p_{j-1,n}, p_{j,n}) \le \frac{\rho_0}{2}$ $(p_{m+1,n} := p_{0,n},\ j = 1, \cdots, m+1)$

Observe that the lengths of γ_n are bounded as they constitute a minimizing sequence. Therefore, we may assume that m is <u>independent</u> of n.

$$
\begin{array}{ccccc}
p_{0,1}, & p_{1,1}, & p_{2,1}, & \cdots & p_{m,1} \\
p_{0,2}, & p_{1,2}, & p_{2,2}, & \cdots & p_{m,2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{0,n}, & p_{1,n} & p_{2,n} & \cdots & p_{m,n} \\
\vdots & \vdots & \vdots & & \vdots \\
\downarrow & \downarrow & \downarrow & & \downarrow \\
p_0 & p_1 & p_2 & \cdots & p_m
\end{array}
$$

By a diagonal argument, up to subsequence,

Recall that the geodesic between $p_{j-1,n}$ and $p_{j,n}$ depends conti. on its endpoints, and hence converges to the shortest arc between p_{j-1} and p_j, ~~which~~ (shortest arc is geodesic)

Those shortest geodesic arcs yields a closed curve γ.

By Lemma 4.1, γ is homotopic to γ_n, and

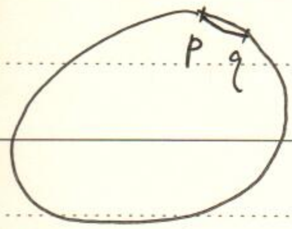
$$\mathrm{Length}(\gamma) = \lim_{n\to\infty} \mathrm{Length}(\gamma_n).$$

Recall (γ_n) is a minimizing sequence for the length in their homotopy class.

Therefore, γ is a shortest curve in this homotopy class.

<u>Claim</u>: γ is a geodesic.

Otherwise, there would exist points p and q on γ for which one of the two arcs of γ between p and q would have length at most $\frac{\rho_0}{2}$, but would not be geodesic.



Then this arc $\overline{pq}$ can be shortened by replacing it by the shortest geodesic between p and q. Denote this new curve as $\tilde{\gamma}$. We have

$$d(\gamma(t), \tilde{\gamma}(t)) \leq \rho_0, \ \forall t \in S^1.$$

Lemma 4.1 $\Rightarrow$ γ and $\tilde{\gamma}$ are homotopic. This contradicts to the minimizing property of γ. $\square$

γ is the desired closed geodesic. $\square$

<u>Remark</u>: If the compact Rie. mfld M is simply-connected, the above argument leads to the trivial closed geodesic: a pt.

Now, we discuss the existence of shortest geodesics in a given homotopic class of curves with fixed initial and end points in a <u>complete</u> Rie. manifold.

<u>Theorem 4.2</u>. Let (M, g_M) be a complete, connected Rie. mfld, and $p, q \in M$. Every homotopy class of paths from p to q contains a geodesic γ that minimizes ~~length~~ among all admissible curves in the same homotopy class.

The idea to prove Theorem 4.2 is first going to the universal covering manifold $\tilde{M}$ of M. In $\tilde{M}$, ~~between~~ curves connecting corresponding ~~two~~ points $\tilde{p}$ and $\tilde{q}$ only have one homotopy class. For that purpose, we need first show that M is complete $\Rightarrow$ $\tilde{M}$ is complete.

Recall. A covering map is a surjective continuous map

$$\pi : \tilde{M} \longrightarrow M$$

between connected and locally-path-connected topological spaces, for which each point of M has connected neighborhood U that is evenly covered, meaning that each ~~point of M has connected neighborhood U that is~~ connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π.

It is called a smooth covering map if $\tilde{M}$ and M are smooth mflds and each component $\pi^{-1}(U)$ is mapped diffeomorphically onto U.

Any Riemannian metric on M induceds a Rie. metric on $\tilde{M}$. This makes π into a Riemannian covering. In particular, π is a local isometry.

Lemma 4.2. Suppose $\tilde{M}$ and M are connected Rie. mflds, and $\pi : \tilde{M} \to M$ is a Riemannian covering map. If M is complete, then $\tilde{M}$ is also complete.

Proof. Let $\tilde{p} \in \tilde{M}$ and $\tilde{v} \in T_{\tilde{p}}\tilde{M}$ be arbitrary, and let $p = \pi(\tilde{p})$, and $v = d\pi(\tilde{p})(\tilde{v})$.

Completeness of M implies that the geodesic γ with

$$\gamma(0)=p \text{ and } \dot\gamma(0)=v$$

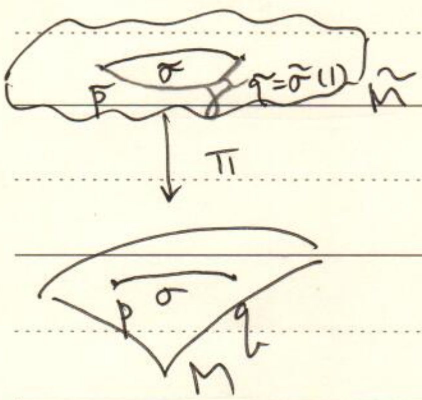
is defined for all $t\in\mathbb{R}$. ~~then~~

(Recall a fundamental property of covering maps is the path-lifting property: If $\pi:\tilde M\to M$ is a covering map, then every continuous path $\gamma: I\to M$ lifts to a path $\tilde\gamma$ in $\tilde M$ s.t. $\pi\circ\tilde\gamma=\gamma$.)

Here the lifts $\tilde\gamma$ of γ starting at $\tilde p$ with the initial tangent vector $\tilde v$. Since π is a local isometry, we know $\tilde\gamma$ is a geodesic. Since γ is defined for all $t\in\mathbb{R}$, so does $\tilde\gamma$. This proves the completeness of $\tilde M$. □

Proof of Thm 4.2.

Consider the universal covering $\pi:\tilde M\to M$ of M, endowed with the induced metric $\tilde g=\pi^* g$.



Given $p,q\in M$ and a path

$$\sigma:[0,1]\to M$$

from p to q.

Choose a $\tilde p\in\pi^{-1}(p)$, and let

$$\tilde\sigma:[0,1]\to\tilde M$$

be the lift of σ starting with $\tilde p$, and set $\tilde q=\tilde\sigma(1)$.

By Hopf-Rinow, and the fact $\tilde M$ is complete, there exists a minimizing $\tilde g$-geodesic $\tilde\gamma$ from $\tilde p$ to $\tilde q$.

Because π is a local isometry, $\gamma:=\pi\circ\tilde\gamma$ is a geodesic from p to q in M.

Since $\tilde{M}$ is simply connected, we have $\tilde{\sigma}$ and $\tilde{\gamma}$ are homotopic. Hence σ and γ are also homotopic. That is γ is a geodesic in the homotopy class $[\sigma]$.
If γ_1 is any other admissible curve from p to q in the homotopy class $[\sigma]$, then by the monodromy theorem, its lifts $\tilde{\gamma}_1$ starting at $\tilde{p}$ also ends at $\tilde{q}$, (and $\tilde{\gamma}_1$ and $\tilde{\sigma}$ are homotopic, which is trivial in a simply connected space)
In $\tilde{M}$, we know $\text{Length}(\tilde{\gamma}) \leq \text{Length}(\tilde{\gamma}_1)$

$\Rightarrow \quad \text{Length}(\gamma) \leq \text{Length}(\gamma_1)$ $\square$

We'd like take this chance to discuss further about Rie. covering map.

<u>Theorem 4.3</u>. Let $(\tilde{M}, \tilde{g})$ and (M, g) are connected Rie. mflds with $\tilde{M}$ complete, and $\pi : \tilde{M} \to M$ is a local isometry. Then M is complete and π is a Riemannian covering map.
(Used later in Cartan-Hardamard, see Lemma 3 on page 3/3 in chap(V))

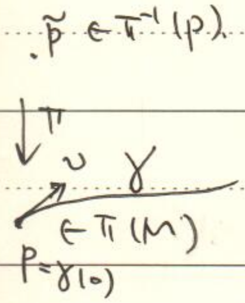
<u>Corollary 4.1</u>. Suppose $\tilde{M}$ and M are connected Rie. mflds. and $\pi : \tilde{M} \to M$ is a Rie. covering map.
Then M is complete iff $\tilde{M}$ is complete.
<u>Proof</u>. Combination of Thm 4.3 & Lemma 4.2.

<u>Proof of Thm 4.3</u>:
- <u>Path-lifting property for geodesic of π</u> a local isometry.

Let $p \in \pi(\tilde{M})$, and $\tilde{p} \in \pi^{-1}(p)$.

let $\gamma: I \to M$ be a geodesic

with $p = \gamma(0)$, $v = \dot{\gamma}(0)$.



Let $\tilde{v} := \left(d\pi(\tilde{p})\right)^{-1}(v) \in T_{\tilde{p}}\tilde{M}$

(π local isometry insures $d\pi(\tilde{p})$ is a linear isometry).

Let $\tilde{\gamma}$ be the geodesic in $\tilde{M}$ with initial pt $\tilde{p}$ and initial tangent vector $\tilde{v}$. Since $\tilde{M}$ is complete, $\tilde{\gamma}$ is defined on all of $\mathbb{R}$.

Since π is a local isometry, $\pi \circ \tilde{\gamma}$ is a geodesic with initial pt p and initial tangent vector v. $\Rightarrow$

$$\pi \circ \tilde{\gamma} = \gamma \quad \text{on } I.$$

So $\tilde{\gamma}|_I$ is a lift of γ starting at $\tilde{p}$.

- M is complete: Let $p \in \pi(M)$, $\gamma: I \to M$ be any geodesic starting at p. then γ has a lift $\tilde{\gamma}: I \to \tilde{M}$. Since $\tilde{M}$ is complete, $\pi \circ \tilde{\gamma}$ is a geodesic defined on all $\mathbb{R}$ and coincides with γ on I. That is γ extends to all of $\mathbb{R}$

Thus M is complete by Hopf - Rinow.

- π is surjective.

$\forall \tilde{p} \in \tilde{M}$, write $p = \pi(\tilde{p})$.

let $q \in M$ be arbitrary. M complete $\xRightarrow{H\text{-}R}$ $\exists$ of a minimizing geodesic γ from p to q.

Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{p}$, and $r = d(p, q)$.

we have $\pi(\tilde{\gamma}(r)) = \gamma(r) = q$. So $q \in \pi(\tilde{M})$.

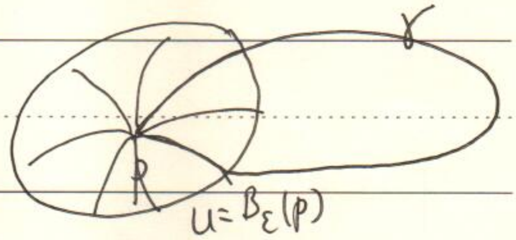
- every pt of M has a neighborhood U that is <u>evenly covered</u>.

Let $p \in M$, let $U = B_\varepsilon(p)$ be a geodesic ball (normal ball) centered at p, $\varepsilon < \mathrm{inj}(p)$.

Write $\pi^{-1}(p) = \{\tilde{p}_\alpha\}_{\alpha \in A}$ & for each α write $\tilde{U}_\alpha$ be the metric ball of radius ε around $\tilde{p}_\alpha$.

<u>Claim</u> $\tilde{U}_\alpha \cap \tilde{U}_\beta = \phi$, $\forall \alpha \neq \beta$.

<u>Proof</u> $\forall \alpha \neq \beta$, there exists a minimizing geodesic $\tilde{\gamma}$ from $\tilde{p}_\alpha$ to $\tilde{p}_\beta$ because $\tilde{M}$ is complete.

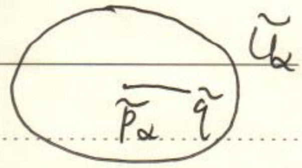


The projective curve $\gamma := \pi \circ \tilde{\gamma}$ is a geodesic that starts and end at p, whose length is the same as that of $\tilde{\gamma}$.

Such a geodesic must leave U and reenter it.

Since all geodesics passing through p are radial geodesics, we have $\underset{\text{Length}(\tilde{\gamma})}{\text{Length}(\gamma)} \geq 2\varepsilon$. $\Rightarrow d_{\tilde{g}}(\tilde{p}_\alpha, \tilde{p}_\beta) \geq 2\varepsilon$. $\Rightarrow U_\alpha \cap U_\beta = \phi$ $\square$

<u>Claim</u> $\pi^{-1}(U) = \bigcup_\alpha \tilde{U}_\alpha$.



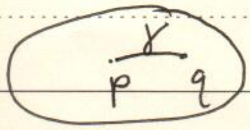
<u>Proof</u>: $\forall \tilde{q} \in \tilde{U}_\alpha$ for some α,

there is a geodesic $\tilde{\gamma}$ of length $< \varepsilon$ from $\tilde{p}_\alpha$ to $\tilde{q}$

Then $\pi \circ \tilde{\gamma}$ is a geodesic of the same length from p to $\pi(\tilde{q})$, showing that $\pi(\tilde{q}) \in U = B_\varepsilon(p)$. i.e. $\bigcup_\alpha \tilde{U}_\alpha \subseteq \pi^{-1}(U)$.

$\forall \tilde{q} \in \pi^{-1}(U)$, we set $q = \pi(\tilde{q})$. That is, $q \in U$.

So there is a minimizing radial geodesic γ in U from q to p, and $r = d_g(p, q) < \varepsilon$.



Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{q}$.

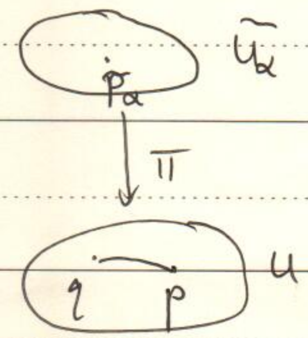
It follows that

$$\pi(\tilde{\gamma}(r)) = \gamma(r) = p.$$

Therefore $\tilde{\gamma}(r) = \tilde{p}_\alpha$ for some α, and $d_{\tilde{g}}(\tilde{q}, \tilde{p}_\alpha) \leq \text{Length}(\tilde{\gamma}) = r < \varepsilon$.

So $\tilde{q} \in \tilde{U}_\alpha$.

- It remains to show that $\pi : \tilde{U}_\alpha \to U$ is a diffeomorphism for each α. It is certainly a local diffeomorphism. It is bijective: we can construct the inverse explicitly: It sends each radial geodesic starting at p to its lift starting at $\tilde{p}_\alpha$.



This completes the proof. □.