Proof of Morse index theorem.
Recall we have \( I(f^T, f^T) = \int_a^b (\dot{f})^2 \, dt \geq 0, \quad \text{"''\"} \Rightarrow f \equiv 0. \)
\[
f(a) = f(b) = 0
\]
\[
I(f^T, U) = 0, \quad \forall U \in \mathcal{V}_0(a, b), \langle U, f \rangle = 0
\]
So we can restrict ourselves to the subspace
\[
\mathcal{V}_0^1 (a, b) := \{ x \in \mathcal{V}_0(a, b) \mid \langle x, f \rangle = 0 \}
\]
when studying index and nullity of \( f \).

For simplicity, let's take \( (a, b) = (0, 1) \).
Firstly, we show \( \text{ind}(f) < \infty \).

Consider we first explain that we can find a finite-dim subspace \( T \) of \( \mathcal{V}_0^1 (0, 1) \) s.t. the index, nullity of \( I \) do not change when restricting to \( T \).

By considering the open covers of \( f|_{[0, 1]} \) by the totally normal neighborhood of each \( f(t_i), t_i \in \{0, 1\} \), we can find a finite subdivision, \( 0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1 \) such that \( f|_{[t_i, t_{i+1}]} \) lies in a neighborhood \( U_i \) totally normal.

In particular, \( f|_{[t_i, t_{i+1}]} \) contains no conjugate point for each \( i \).
Define

\[ T_i := T_{i, i} := \left\{ X \in \mathcal{V}_0^1 : X \text{ is Jacobian along each } x \mid (t_i, t_i+1) \right\}, \quad i = 0, \ldots, k \]

\[ T_2 := T_2(1) = \left\{ X \in \mathcal{V}_0^1 : X(t_1) = 0, \forall i = 0, \ldots, k+1 \right\} \]

Lemma A. We have (i) \[ T_0^+ (0, 1) = T_1 \oplus T_2 \]

(ii) \[ I(T_1, T_2) = 0 \]

(iii) \[ I_{T_2} \text{ is positive definite.} \]

Proof. Consider the map

\[ \varphi : T_1 \rightarrow T_{0, 0} \oplus \cdots \oplus T_{0, 0} M \]

\[ J \rightarrow (J(t_1), \ldots, J(t_k)) \]

Clearly, this is a linear map and 1-1.

(Since on \[ X \mid (t_i, t_i+1), J \text{ is uniquely determined by } J(t_i) \text{ and } J(t_i+1). \]

Therefore \( \varphi \) is a linear isometry. In particular, \( \dim T_i = nk \).

Given any \( X \in \mathcal{V}_0^1 (0, 1) \). Let \( J_x := \varphi^{-1} (X(t_1), \ldots, X(t_k)) \).

Then we have \( J_x \in T_1, X - J_x \in T_2 \).

Moreover, \( T_1 \cap T_2 = \{0\} \) since \( J(t_1) = 0 \Rightarrow J(t_{i+1}) = 0 \) on \( X \mid [t_i, t_{i+1}] \).

This shows (i) \( T_0^+ (0, 1) = T_1 \oplus T_2 \).

For (ii), we have \( \forall X_1 \in T_1, X_2 \in T_2 \)

\[ I(X_1, X_2) = \left< \nabla_{T_1} X_1, X_2 \right>_0 - \sum_{i=0}^{k} \left< \nabla_{T_i} X_1 - \nabla_{T_i} X_2, X_i \right> \]

\[ = 0. \]

For (iii). For any \( X \in T_2 \),

\[ I(X, X) = \sum_{i=0}^{k} I_{T_i}^{+1} (X(t_i), X(t_i)) > 0. \]

\( \square \)
By Lemma A, we obtain immediately,
\[ \text{ind}(\phi) \leq \dim(T_1) < \infty. \]
and the index \( \text{ind} \), nullity \( N \) of \( T_1 \) equal to \( \text{ind}(\phi), N(\phi) \) respectively.

**Lemma B**

(i) \( \forall \tau \in (0, 1), \exists \delta > 0, \text{ s.t. } \forall \varepsilon \in [0, \delta], \text{ we have} \)
\[ \text{ind}(\tau) = \text{ind}(\tau + \varepsilon). \]

(ii) \( \forall \tau \in [0, 1), \exists \delta > 0, \text{ s.t. } \forall \varepsilon \in [0, \delta], \text{ we have} \)
\[ \text{ind}(\tau + \varepsilon) = \text{ind}(\tau) + N(\tau). \]

**Proof.** For given \( \tau \), we can assume the division. We choose previously a subsequence \( (\tau_i) \) such that \( \tau \in (\tau_i, \tau_{i+1}) \).

Denote \( T_1(T) := \{ X \in T^1(0, 1), X|_{[t_i, t_{i+1}]} \text{ is Jacobian, } i = 0, \ldots, d \} \)
\[ \text{X|}_{[t_i, t_{i+1}]} \text{ is also Jacobian.} \]

Similarly, consider
\[ g^T : T_1(T) \to T^{g(T)} \oplus \cdots \oplus T^{g(T_d)} \]
\[ X \mapsto (X(g(T_1)), \ldots, X(g(T_d))) \]
is a linear isometry.

\( I_{\tau} \big|_{T_{1}(\tau)} \) can be considered as the a bilinear quadratic form over \( T^{g(T_1)} \oplus \cdots \oplus T^{g(T_d)} \) in the sense
\[ I_{\tau}(x, y) := I_{\tau}((g(T)^{-1}(x), (g(T)^{-1}(y)) \right)
\]
Note \( \text{ind}(\tau) \) is the index of \( I_{\tau} \big|_{T_{1}(\tau)} \).
\[ H_{\text{erm}} \forall x, y \in T_1(t), \quad \text{define } x_i = x_{[t_i, t_{i+1}]} \quad x_j = x_{[t_j, t_{j+1}]} \]
\[
I_0^\epsilon(x, y) = \sum_{i=0}^{2n} \langle \nabla_t x_i, y \rangle_{t_i}^2 - \langle \nabla_t x_j(t_j), y(t_j) \rangle^2.
\]

For given \( \epsilon \), given division \( 0 = t_0 < t_1 < \ldots < t_j < t_{j+1} < \ldots < t_{k+1} = 1 \),
the \( T_1(t) \equiv T_{\gamma(t_0)}M \oplus \ldots \oplus T_{\gamma(t_j)}M \) is a fixed vector space. So when \( t \) changes, only \( \langle \nabla_t x_j(t_j), y(t_j) \rangle \)
change in \( I_0^\epsilon(x, y) \), where \( x = (x_1, \ldots, x^j), \quad y = (y_1, \ldots, y^j) \).

\( x_j \) and \( y \) are Jacobi fields on \( \gamma(t_j, t) \) with
\[
x_j(t_j) = \gamma(t_j), \quad x_j(0) = 0
\]
\[
y(t_j) = y(t_j), \quad y(0) = 0.
\]

By construction, \( x_{[t_j, t_{j+1}]} \subseteq U_j \) a totally normal neighborhood.
Hence any geodesics \( \gamma \) lying inside \( U_j \) depends smoothly on their endpoints.

Since Jacobi fields are variational vector fields, \( \gamma \) depends in a smooth way.
We have \( x_j, y_{[t_j, t_{j+1}]} \) also depends continuously on endpoints \( x(t_j), \) and \( y(t) \).

Therefore, \( \langle \nabla_t x_j(t_j), y(t_j) \rangle \) depend are smooth w.r.t. \( t \).
That is \( I_0^\epsilon(x, y) \) is a continuous w.r.t. \( t \) for given \( x, y \).

So for \( x \in \oplus T_1(t)M \), \( I_0^\epsilon(x, x) < 0 \) implies
\[
\exists \delta > 0 \text{ s.t. } I_0^{\epsilon + \delta}(x, x) < 0, \forall \epsilon \in [0, \delta].
\]

This tells \( \text{ind}(\epsilon) \geq \text{ind}(\epsilon) \).

(Explanation, \( I_0^\epsilon : T_1(0) \times T_1(t) \to \mathbb{R} \) quadratic form.)
By linear algebra theory, the linear space $T_1(\tau)$ can be decomposed into:

- maximal positive definite subspace
- maximal negative definite subspace
- null space $\{ X \in T_1(\tau) \mid I(X, Y) = 0, \forall Y \in T_1(\tau) \}$

$n_\parallel = \dim (\oplus_{\tau' \in \tau} T_{\tau' + \tau}(\mathbb{R})) = \text{ind}_+ (\mathbb{R}) + \text{ind}_- (\mathbb{R}) + N(\mathbb{R})$

$n_\parallel = \dim (\oplus_{\tau' \in \tau} T_{\tau' + \tau}(\mathbb{R})) = \text{ind}_+ (\mathbb{R}) + \text{ind}_- (\mathbb{R}) + N(\mathbb{R})$

Using $\text{ind}_+ (\tau) = n_\parallel - \text{ind}_- (\mathbb{R}) - N(\mathbb{R})$, we derive from (2) that:

$n_\parallel - \text{ind}_- (\tau \pm \epsilon) - N(\mathbb{R}) \leq n_\parallel - \text{ind}_- (\mathbb{R}) - N(\mathbb{R})$

i.e.

$\text{ind}_- (\tau \pm \epsilon) \leq \text{ind}_- (\mathbb{R}) + N(\mathbb{R})$

Combining (1) and (3) gives:

$\text{ind}_- (\mathbb{R}) \leq \text{ind}_- (\tau \pm \epsilon) \leq \text{ind}_- (\mathbb{R}) + N(\mathbb{R})$.

For any $j_0, j_1 < j < \tau < j_{j+1}$,

$\forall x \in \oplus_{\tau' \in \tau} T_{\tau' + \tau}(\mathbb{R}) \mid M$, we have

$I^x_0 (x, x) - I^x_0 (-x, x) = I^x_{j_0} (X_{j_0}, X_{j_0}) - I^x_{j_0} (X_{j_0}, X_{j_0})$

where $X_{j_0}$ is the Jacobian field with $X_{j_0} (t_j) = X_{j_0}$, $X_{j_0} (t_j) = 0$

By minimizing property of Jacobian field, we have

$I (X_{j_0}, X_{j_0}) \leq I (X_{j_0}, X_{j_0})$

and $\iff$ holds iff $X_{j_0} = X_{j_0} \iff X_{j_0} = 0$.
That is, $I_0^c(x,x) \leq I_0^3(x,x)$, and "=" holds if $x_j = 0$.

Observe that $x_j = (g_3^3)^{-1}(x) \cdot t_j = 0$. Since otherwise, we have $(g_3^3)^{-1}(x) \cdot t_j, s_j \equiv 0 \pmod{\mathbb{F}_3}$ contains no conjugate pt.

Hence

(i) $I_0^3(x,x) < 0 \Rightarrow I_0^c(x,x) < 0$

(ii) Let $x$ be in the null space of $I_0^3$. Then $(g_3^3)^{-1}(x) \in V_+^c(0,3)$ is a Jacobi field vanishing at $t = 0$ and $3$.

Observe that $x_j = (g_3^3)^{-1}(x) \cdot t_j \neq 0$. Since otherwise, we have $(g_3^3)^{-1}(x) \cdot t_j, s_j \equiv 0 \pmod{\mathbb{F}_3}$ contains no conjugate pt.

and therefore $(g_3^3)^{-1}(x) \equiv 0 \Rightarrow x \equiv 0$, contradicting to $x \neq 0$.

Therefore, we have $I_0^c(x,x) < I_0^3(x,x) = 0$, $\forall x \in$ the null space of $I_0^3$.

In conclusion, $1^1 + 3^1$ implies

\[ \text{ind}(1) \geq \text{ind}(3) + M^3 \]  

(5)

We have

\[ \text{ind}(2) \leq \text{ind}(1) \leq N(1-3-3) \leq \text{ind}(2) - N(1-3) \leq \text{ind}(2). = \text{ind}(2) = \text{ind}(2) \]  

(4)

\[ \text{ind}(2+3) \leq \text{ind}(2) + N(2+3) \leq \text{ind}(2+3). \Rightarrow \text{ind}(2+3) = \text{ind}(2+3) = \text{ind}(2+3) \]  

Proof of Main Lemma $\Rightarrow$ i. If $\gamma(2)$ is not a conjugate point of $\gamma(1)$, $\exists \delta > 0$ s.t. $\text{ind}(t) \mid [1-2+\delta] \text{ is constant}$.

ii. If $\gamma(2)$ is conjugate to $\gamma(1)$, $\exists \delta > 0$ s.t. $\text{ind}(t) \mid [1-2+\delta] \text{ is constant}$ and $\text{ind}(t) \mid [2+\delta] \text{ is also constant}.$
And the jump size of \( \text{ind}(t) \) at \( t = 2 \) is \( N(2) \).

So when \( t \) changes from 0 to 1, \( \text{ind}(t) \) changes from 0, and jump \( N(t)/2 \) wherever \( t \) is a conjugate value.

Since \( \text{ind}(1) < \infty \), we know this jump can only happen finitely many times.

\[ \Rightarrow \quad \text{ind}(1) = \sum_{1 = 2^0}^{N(2)} \]