Proof of Morse index theorem

Recall we have $I(fT, fT) = \int_a^b (\dot{f})^2 \, dt \geq 0$, "=" $\Leftrightarrow f \equiv 0$.

$f(a) = f(b) = 0$

$I(fT, U) = 0, \ \forall \ U \in \mathcal{V}_0(a,b), \ \langle U, T \rangle = 0$

So we can restrict ourself to the subspace

$$\mathcal{V}_0^{\perp}(a,b) := \{ X \in \mathcal{V}_0(a,b) \mid \langle X, T \rangle = 0 \}.$$

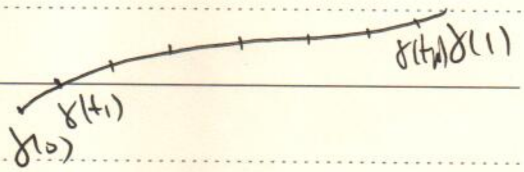
when studying index and ~~a~~ nullity of γ.

For simplicity, let's take $(a,b) = (0,1)$.

Firstly, we show $\mathrm{ind}(\gamma) < \infty$.

~~Consider the~~ We first explain that ~~there~~ we can find a finite-dim subspace T_1 of $\mathcal{V}_0^{\perp}(0,1)$ s.t. the index, nullity of I ~~doesnot~~ do not change when restricting to T_1.

By considering the open covers of $\gamma|_{[0,1]}$ by the totally normal neighborhood of each $\gamma(t)$, $t \in [0,1]$, we can



find a finite subdivision, $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$

such that $\gamma|_{[t_i, t_{i+1}]}$ lies in a totally normal neighborhood U_i.

In particular, $\gamma|_{[t_i, t_{i+1}]}$ contains no conjugate point for each i.

Define :

$T_1 := T_1(1) := \{ X \in \mathcal{V}_0^\perp : X$ is Jacobian along each $\gamma|_{[t_i, t_{i+1}]}$ $\forall\, i = 0, \dots, k \}$

piecewise Jacobian.

$T_2 := T_2(1) = \{ X \in \mathcal{V}_0^\perp : X(t_i) = 0, \forall\, i = 0, \dots, k+1 \}$

Lemma A. We have (i) $\mathcal{V}_0^\perp(0,1) = T_1 \oplus T_2$.

(ii) $I(T_1, T_2) = 0$

(iii) $I|_{T_2}$ is positive definite.

Proof : Consider the map

$$\varphi : T_1 \longrightarrow T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_k)}M.$$

$$J \longmapsto (J(t_1), \dots, J(t_k))$$

Clearly, this is a linear map, and 1-1.

(Since on $\gamma|_{[t_i, t_{i+1}]}$, $\leftarrow$ no conj. J is uniquely determined by $J(t_i)$ and $J(t_{i+1})$.)

Therefore φ is a linear isometry. In particular, $\dim T_1 = nk < \infty$.

Given any $X \in \mathcal{V}_0^\perp(0,1)$. Let $J_X := \varphi^{-1}(X(t_1), \dots, X(t_k))$

Then we have $J_X \in T_1$, $X - J_X \in T_2$.

Moreover, $T_1 \cap T_2 = \{0\}$ since $J(t_i) = 0 = J(t_{i+1}) \Rightarrow J \equiv 0$ on $\gamma|_{[t_i, t_{i+1}]}$.

This shows (i) $\mathcal{V}_0^\perp(0,1) = T_1 \oplus T_2$.

For (ii), we have $\forall\, X_1 \in T_1$, $X_2 \in T_2$

$$I(X_1, X_2) = \langle \nabla_T X_1, X_2 \rangle \Big|_0^1 - \sum_{j=1}^{k} \langle \nabla_{T(t_j^+)} X_1 - \nabla_{T(t_j^-)} X_1, X_2 \rangle$$

$$= 0.$$

For (iii). For any $X \in T_2$,

$$I(X, X) = \sum_{i=0}^{k} I_{t_i}^{t_{i+1}}(X_i, X_i) > 0. \qquad \square$$

By Lemma A, we obtain immediately,

$$\operatorname{ind}(\gamma) \le \dim(T_1) < \infty .$$

and the index ~~of~~, nullity of $I|_{T_1}$ equal to $\operatorname{ind}(\gamma)$, $N(\gamma)$ respectively.

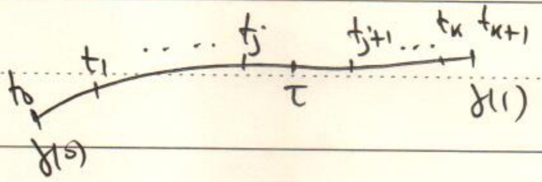
<u>Lemma B</u> (i) $\forall\, \tau \in (0,1]$, $\exists\, \delta > 0$, s.t. $\forall\, \varepsilon \in [0,\delta]$ we have

$$\cancel{\theta(\tau-\varepsilon) =} \operatorname{ind}(\tau-\varepsilon) = \operatorname{ind}(\tau).$$

(ii) $\forall\, \tau \in [0,1)$, $\exists\, \delta > 0$ s.t. $\forall\, \varepsilon \in [0,\delta]$, we have

$$\operatorname{ind}(\tau+\varepsilon) = \operatorname{ind}(\tau) + N(\tau).$$

<u>Proof</u>: For given τ, we can assume the division we choose previously has the property that $\tau \in (t_j, t_{j+1})$.



Denote
$$T_1(\tau) := \left\{ X \in \mathcal{V}_0^{\perp}(0,1),\ \begin{array}{l} X|_{[t_i, t_{i+1}]} \text{ is Jacobian}, \ i = 0, \cdots, j-1 \\ X|_{[t_j, \tau]} \text{ is also Jacobian} \end{array} \right\}$$

Similarly, consider

$$\varphi^\tau : T_1(\tau) \longrightarrow T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_j)}M$$
$$X \longmapsto (X(t_1), \cdots, X(t_j))$$

is a linear isometry.

$I_0^\tau|_{T_1(\tau)}$ can be considered as ~~the~~ a ~~bilinear~~ qudadratic form over $T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_j)}M$ in the sense

$$I_0^\tau(x, y) := I_0^\tau\left((\varphi^\tau)^{-1}(x), (\varphi^\tau)^{-1}(y)\right).$$

Note $\operatorname{ind}(\tau)$ is the index of $I_0^\tau|_{T_1(\tau)}$.

Hence $\forall X, Y \in T_1(\tau)$, denote $X_i = X|_{[t_i, t_{i+1}]}$, $X_j = X|_{[t_j, \tau]}$

$$I_0^\tau(X,Y) = \sum_{i=0}^{j-1} \langle \nabla_T X_i, Y \rangle \Big|_{t_i}^{t_{i+1}} - \langle \nabla_T \dot{X_j}(t_j), Y(t_j) \rangle.$$

For ~~given X, Y~~, given divison $0 = t_0 < t_1 < \cdots < t_j < t_{j+1} < \cdots < t_{k+1} = 1$, the $T_1(\tau) \cong T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_j)}M$ is a fixed vector space. So when τ changes, only $\langle \nabla_T \dot{X}_j(t_j), Y(t_j) \rangle$ change in $I_0^\tau(x,y)$, where $x = (x^1, \ldots x^j)$, $y = (y^1, \ldots, y^j) \in \bigoplus_{i=1}^{j} T_{\gamma(t_i)}M$

X_j and Y are Jacobi field on $[t_j, \tau)$ with

$$X_j(t_j) = x^j, \quad X_j(\tau) = 0$$

$$Y(t_j) = y^j, \quad Y(\tau) = 0$$

By construction, $\gamma|_{[t_j, t_{j+1}]} \subset U_j$ a totally normal neighborhood.

~~Hence any geodesics lying inside U_j depends smoothly on their endpoints.~~

~~Since Jacobi fields are variational vector fields of geodesic~~ variations, we have X_j, $Y|_{[t_j, \tau]}$ also depends continuously on endpoints $\gamma(t_j)$, and $\gamma(\tau)$.

Therefore $-\langle \nabla_T X_j(t_j), Y(t_j) \rangle$ ~~depend~~ are smooth w.r.t. τ.

That is $I_0^\tau(x,y)$ is a continuous w.r.t. τ for given x, y.

So for $x \in \bigoplus_{i=1}^{j} T_{\gamma(t_i)}M$, $I_0^\tau(x,x) \underset{(>0)}{<0}$ implies

$$\exists\, \delta > 0 \text{ s.t. } I_0^{\tau \pm \varepsilon}(x,x) \underset{(>0)}{<0}, \quad \forall \varepsilon \in [0, \delta].$$

This tells $\quad \operatorname{ind}(\tau \pm \varepsilon) \geq \operatorname{ind}(\tau) \quad (1)$

and $\quad \operatorname{ind}_+(\tau \pm \varepsilon) \geq \operatorname{ind}_+(\tau). \quad (2)$

(~~Explanation~~ Notation: $I_0^\tau : T_1(\tau) \times T_1(\tau) \to \mathbb{R}$ quadratic form

By linear algebraic theory, the linear space $T_1(\tau)$ can be ~~det~~ decomposed into

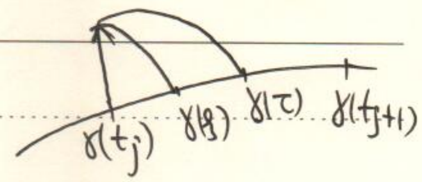
subspaces	dimension
· maximal positive definite subspace	$ind_+(\gamma)$
· maximal negative definite subspace	$ind(\gamma)$
· null space $\{X \in T_1(\tau) \mid I(X,Y)=0, \forall Y \in T_1(\tau)\}$	$N(\gamma)$

$$nj = \dim\Big(\bigoplus_{i=1}^{j} T_{\gamma(t_i)}M\Big) = ind_+(\tau) + ind(\tau) + N(\tau)$$

Using $ind_+(\tau) = nj - ind(\tau) - N(\tau)$, we derive from (2) that
$$nj - ind(\tau\pm\varepsilon) - N(\tau\pm\varepsilon) \geq nj - ind(\tau) - N(\tau)$$
i.e.
$$\begin{aligned} ind(\tau\pm\varepsilon) &\leq ind(\tau) + N(\tau) - N(\tau\pm\varepsilon) \\ &\leq ind(\tau) + N(\tau). \qquad (3)\end{aligned}$$

Combining (1) and (3) gives $ind(\tau) \leq ind(\tau\pm\varepsilon) \leq ind(\tau) + N(\tau)$ (4).

For any ξ, $t_j < \xi < \tau < t_{j+1}$,



$\forall\, x \in \bigoplus_{i=1}^{j} T_{\gamma(t_i)}M$, we have
$$I_0^{\xi}(x,x) - I_0^{\tau}(x,x) = I_{t_j}^{\xi}(X_{j,\xi}, X_{j,\xi}) - I_{t_j}^{\tau}(X_{j,\tau}, X_{j,\tau})$$
where $X_{j,\xi}$ is the Jacobi field with $X_{j,\xi}(t_j) = x_j$, $X_{j,\xi}(\xi) = 0$
$X_{j,\tau}$ · · · " " $X_{j,\tau}(t_j) = x_j$, $X_{j,\tau}(\tau) = 0$

By Minimizing property of Jacobian field, we have
$$I(X_{j,\tau}, X_{j,\tau}) \leq I(X_{j,\xi}, X_{j,\xi})$$
and "=" holds iff $X_{j,\tau} = X_{j,\xi} \Leftrightarrow x_j = 0$

~~Observe (i)~~

That is

$$I_0^{\tau}(x,x) \le I_0^{\xi}(x,x) \quad , \text{ and "=" holds iff } x_{j,\tau}=0.$$

~~Observe~~ Hence

(i) $I_0^{\xi}(x,x) < 0 \Rightarrow I_0^{\tau}(x,x) < 0$

(ii) Let $\underset{\neq 0}{x}$ be in the null space of I_0^{ξ}. Then

$(\varphi^{\xi})^{-1}(x) \in \mathcal{V}_0^{\perp}(0,\xi)$ is a (normal) Jacobi field vanishing at $t=0$ and ξ.

Observe that $x_j = (\varphi^{\xi})^{-1}(x)\,(t_j) \neq 0$. Since otherwise, we have $(\varphi^{\xi})^{-1}(x)\big|_{[t_j,\xi]} \equiv 0$ ($\gamma_{[t_j,\xi]}$ contains no conjugate pt)

and therefore $(\varphi^{\xi})^{-1}(x) \equiv 0 \Rightarrow x \equiv 0$, contradicting to $x \neq 0$.

Therefore, we have

$$I_0^{\tau}(x,x) < I_0^{\xi}(x,x) = 0, \quad \forall x \in \text{the null space of } I_0^{\xi}$$

In conclusion, (i) + (ii) implies

$$\mathrm{ind}(\tau) \ge \mathrm{ind}(\xi) + N(\xi) \qquad (5)$$

We have

$$\mathrm{ind}(\tau) \overset{(4)}{\le} \mathrm{ind}(\tau-\varepsilon) \overset{(5)}{\le} \mathrm{ind}(\tau) - N(\tau-\varepsilon) \le \mathrm{ind}(\tau). \Rightarrow \mathrm{ind}(\tau) = \mathrm{ind}(\tau-\varepsilon)$$

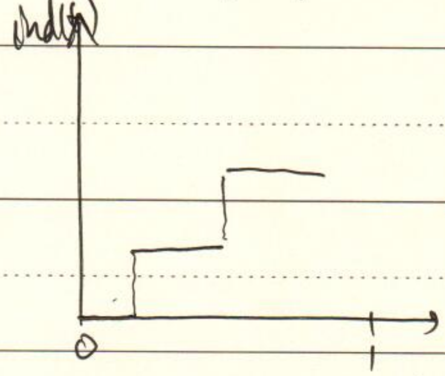
$$\mathrm{ind}(\tau+\varepsilon) \overset{(4)}{\le} \mathrm{ind}(\tau) + N(\tau) \overset{(5)}{\le} \mathrm{ind}(\tau+\varepsilon). \Rightarrow \mathrm{ind}(\tau+\varepsilon) = \mathrm{ind}(\tau) + N(\tau)$$

$\square$

<u>Proof of Morse</u>. Lemma B $\Rightarrow$ · if $\gamma(\tau)$ is not a conjugate point of $\gamma(0)$, $\exists\, \delta>0$ s.t. $\mathrm{ind}(t)\big|_{(\tau-\delta,\tau+\delta)}$ is constant.

· if $\gamma(\tau)$ is conjugate to $\gamma(0)$, $\exists\, \delta>0$ s.t. $\mathrm{ind}(t)\big|_{(\tau-\delta,\tau)}$ is constant and $\mathrm{ind}(t)\big|_{(\tau,\tau+\delta)}$ is also constant.

And the jump size of $\mathrm{ind}(t)$ at $t=\tau$ is $N(\tau)$.



So when t change from 0 to 1, $\mathrm{ind}(t)$ changes from 0, and jump $N(\tau)\geq 1$ whenever τ is a conjugate value. Since $\mathrm{ind}(1)<\infty$, we know this jump can only happen finitely many times.

$$\Rightarrow \quad \mathrm{ind}(1) = \sum_{0<\tau<1} N(\tau). \qquad \square$$