$E^3=\mathbb{R}^3$上微分形式：

回忆混合积 (v_1, v_2, v_3) ~~是~~可以看作一个 ~~3~~ trilinear, alternative

映射. $(v_1, v_2, v_3) = \det \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}$

$\mathrm{sgn}(\sigma)\left(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\right) = \cancel{\mathrm{sgn}(\sigma)}\ (v_1, v_2, v_3)$

$\mathbb{R}^3$上的 1-form 可表为 $\phi = \sum_{i=1}^{3} \phi(U_i)\, dx^i$.

定义. $\mathbb{R}^3$ 的一个 2-form η 是 $\mathbb{R}^3$ 的所有切向量对 (v_p, w_p) 上的实值函数，使得在任意点 p 处 η 是双线性的, alternative,

i.e. $\forall p \in \mathbb{R}^3 \quad \eta(v_p, w_p) = -\eta(w_p, v_p)$.

若对任意光滑向量场 V, W，实值函数

$$\eta(V, W) : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$p \longmapsto \eta(V(p), W(p))$$

是光滑的，则称η为光滑的 2-form.

性质：设η为$\mathbb{R}^3$上一个 2-form，则 η 可表为

$$\eta = \sum_{i<j} \eta(U_i, U_j)\, dx^i \wedge dx^j.$$

其中 $dx^i \wedge dx^j := dx^i \otimes dx^j - dx^j \otimes dx^i$.

$dx^i \otimes dx^j$ ~~给出~~在每点处是双线性映射，也即对任意向量场 V, W, $\quad dx^i \otimes dx^j (V, W) = dx^i(V)\, dx^j(W)$.

证明：易验证 $dx^i \wedge dx^j = -dx^j \wedge dx^i$. 且 $\sum_{i<j} \eta(U_i, U_j)\, dx^i \wedge dx^j$ 是一个 2-form. 对每一对切向量 (v_p, w_p), 记

$$v_p = v^i U_i(p) \quad , w_p = w^j U_j(p).$$

我们有 $\eta(v_p, w_p) = v^i w^j \eta(U_i(p), U_j(p))$

$$=\sum_{i,j}\eta(U_i(p),U_j(p))\,dx^i(v_p)\,dx^j(w_p)$$

$$=\sum_{i<j}\eta(U_i(p),U_j(p))\left(dx^i(v_p)dx^j(w_p)-dx^j(v_p)dx^i(w_p)\right)$$

$$=\sum_{i<j}\eta(U_i(p),U_j(p))\left(dx^i\wedge dx^j\right)(v_p,w_p).\quad\square$$

定义. $\mathbb{R}^3$ 的一个 3-form ψ 是 $\mathbb{R}^3$ 的所有切向量 triple

$$(v_p, w_p, u_p)$$

上的实值函数，使在任意点 p 处，ψ 是 trilinear, alternative.

~~i.e. $\forall p\in\mathbb{R}^3$~~，~~如~~ 如对任意光滑向量场 V, W, U，

实值函数 $\psi(V,W,U):\mathbb{R}^3\to\mathbb{R}$

$$p\longmapsto\psi(V(p),W(p),U(p))$$

是光滑的，则称 ψ 为光滑 3-form.

性质：设 ψ 为 $\mathbb{R}^3$ 上一个 3-form，则 ψ 可表为

$$\psi=\cancel{\sum_{\sigma\in S(3)}\eta(U_\sigma}\ \eta(U_1,U_2,U_3)\,dx^1\wedge dx^2\wedge dx^3$$

其中 $dx^1\wedge dx^2\wedge dx^3=\sum_{\sigma\in S(3)}\operatorname{sgn}(\sigma)\,dx^{\sigma(1)}\otimes dx^{\sigma(2)}\otimes dx^{\sigma(3)}$.

证明：可验证 $\eta(U_1,U_2,U_3)dx^1\wedge dx^2\wedge dx^3$ 是一个 3-form.

对任一组切向量 triple (v_p, w_p, u_p)

$$v_p=v^iU_i(p),\quad w_p=w^iU_i(p),\quad u_p=u^iU_i(p).$$

我们有 $\eta(v_p, w_p, u_p)$

$$=v^iw^ju^k\,\eta(U_i,U_j,U_k)$$

$$=\eta(U_i,U_j,U_k)\,dx^i(v_p)\,dx^j(w_p)\,dx^k(u_p).$$

$$=\sum_{i,j,k}\eta(U_i,U_j,U_k)\,dx^i\otimes dx^j\otimes dx^k\,(v_p,w_p,u_p)$$

$$\underset{\substack{\uparrow \\ \text{alternativity} \\ \text{of } \eta.}}{=} \eta(u_1,u_2,u_3)\, dx^1\wedge dx^2\wedge dx^3(v_p, \omega_p, u_p). \quad \square$$

更关键的是 可以整合很多偏微分运算的 外微分运算.

记 $\Omega_i = \{\mathbb{R}^3$上的光滑 i-form$\}$, $i=0,1,2,3$,

0-form 即函数.

注意没有非平凡4-form, 原因是 $\eta(u_i,u_j,u_k,u_l)=0$.
$1\le i,j,k,l\le 3$.

外微分: $\Omega_0 \xrightarrow{d} \Omega_1 \xrightarrow{d} \Omega_2 \xrightarrow{d} \Omega_3$

(1) $\forall f \in \Omega_0$, $df = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} dx^i$

(2) $\forall \phi \in \Omega_1$, 有 $\phi = \sum_{i=1}^{3} \phi(u_i)\, dx^i$

$$d\phi := \sum_{i=1}^{3} d(\phi(u_i))\wedge dx^i = \sum_{i,j=1}^{3} \frac{\partial(\phi(u_i))}{\partial x^j} dx^j \wedge dx^i$$

(3) $\forall \eta \in \Omega_2$, 有 $\eta = \sum_{i<j} \eta(u_i,u_j)\, dx^i\wedge dx^j$

$$d\eta := \sum_{i<j} \frac{\partial \eta(u_i,u_j)}{\partial x^k} dx^k \wedge dx^i \wedge dx^j.$$

$$= \frac{\partial(\eta(u_1,u_2))}{\partial x^3} dx^3\wedge dx^1\wedge dx^2 + \frac{\partial(\eta(u_1,u_3))}{\partial x^2} dx^2\wedge dx^1\wedge dx^3$$

$$+ \frac{\partial(\eta(u_2,u_3))}{\partial x^1} dx^1\wedge dx^2\wedge dx^3$$

性质: $d^2 = d\circ d = 0$

证明: (1). $\forall f \in \Omega_0$. $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$

$$d(df) = \sum_i d\left(\frac{\partial f}{\partial x^i}\right)\wedge dx^i = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i$$

$$= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \frac{\partial^2 f}{\partial x^j\partial x^i}\right) dx^j\wedge dx^i = 0 \text{ since } \frac{\partial^2 f}{\partial x^i\partial x^j} = \frac{\partial^2 f}{\partial x^j\partial x^i}.$$

(2). $\forall \phi \in \Omega_1$, $\phi = \phi_i dx^i$

$$d\phi = \frac{\partial \phi_i}{\partial x^j} dx^j \wedge dx^i$$

$$d(d\phi) = \frac{\partial^2 \phi_i}{\partial x^j \partial x^k} dx^k \wedge dx^j \wedge dx^i$$

$$= \sum_i \sum_{j<k} \left(\frac{\partial^2 \phi_i}{\partial x^j \partial x^k} - \frac{\partial^2 \phi_i}{\partial x^k \partial x^j} \right) dx^k \wedge dx^j \wedge dx^i = 0. \quad \square$$

如果我们把 1-form, 2-form 分别 1-1 对应于向量场如下：

$$\sum_i f_i dx^i \overset{(1)}{\longleftrightarrow} \sum_i f_i U_i \overset{(2)}{\longleftrightarrow} f_1 dx^2 \wedge dx^3 - f_2 dx^1 \wedge dx^3 + f_3 dx^1 \wedge dx^2$$

则有 $\operatorname{grad} f = \sum_i \frac{\partial f}{\partial x^i} U_i \overset{(1)}{\longleftrightarrow} \sum_i \frac{\partial f}{\partial x^i} dx^i = df$

向量场 $F = \sum_{i=1}^{3} f_i U_i$ 的旋度：

$$F = \sum_i f_i U_i \overset{(1)}{\longleftrightarrow} \phi = \sum_i f_i dx^i$$

$$\operatorname{rot} F \overset{(2)}{\longleftrightarrow} d\phi = \sum_{i,j} \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i$$

$$= \sum_{i<j} \left(\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} \right) dx^i \wedge dx^j$$

向量场 $F = \sum_{i=1}^{3} f_i U_i$ 的散度：

$$F = \sum_i f_i U_i \overset{(2)}{\longleftrightarrow} \eta = f_1 dx^2 \wedge dx^3 - f_2 dx^1 \wedge dx^3 + f_3 dx^1 \wedge dx^2$$

$$d\eta = \frac{\partial f_1}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3 - \frac{\partial f_2}{\partial x^2} dx^2 \wedge dx^1 \wedge dx^3 + \frac{\partial f_3}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2$$

$$= \left(\sum_{i=1}^{3} \frac{\partial f_i}{\partial x^i} \right) dx^1 \wedge dx^2 \wedge dx^3$$

$$= (\operatorname{div} F)\, dx^1 \wedge dx^2 \wedge dx^3.$$

关于 1-form ~~$f \in \Omega_0$~~ $df \in \Omega_1$，$f \in \Omega_0$ 的说明：

$$df(v_p) := \frac{d}{dt}\Big|_{t=0} f(p + t v_p)$$

在 $\mathbb{R}^3$ 中 设 $p = (\underbrace{p_1, p_2, p_3}_{\text{坐标}})$，设 $v_p = \sum_{i=1}^{3} v_i U_i(p)$.

则 $p + t v_p$ 的坐标为 $(p_1 + t v_1, p_2 + t v_2, p_3 + t v_3)$

故而 $\frac{d}{dt}\Big|_{t=0} f(p + t v_p)$

$$= \frac{d}{dt}\Big|_{t=0} f\left(p_1 + t v_1, p_2 + t v_2, p_3 + t v_3\right)$$

$$= \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p) v^i = \langle \operatorname{grad} f(p), v_p \rangle$$

即 $df(v_p) = \langle \operatorname{grad} f(p), v_p \rangle$.

易见，在 p 点处，$df : T_p\mathbb{R}^3 \longrightarrow \mathbb{R}$ 是线性的.