作业3.

1. 设 $f, g \in \Omega_0$, $\phi, \psi \in \Omega_1$. 证明

(1) $d(fg) = (df)g + f\,dg$

(2) $d(f\phi) = df \wedge \phi + f\,d\phi$

(3) $d(\phi\wedge\psi) = d\phi \wedge \psi - \phi \wedge d\psi$

2. 对任意三个 1-forms, $\phi_i := \sum_{j=1}^{3} f_{ij}\,dx^j$, $i=1,2,3$.

回忆 $\phi_1 \wedge \phi_2 \wedge \phi_3 \cancel{= \sum_{j,k,l=1}^{3} f_{1j} f_{2k} f_{3l}\, dx^j \wedge dx^k \wedge dx^l}$.

$$= \left(\sum_{j=1}^{3} f_{1j}\,dx^j\right) \wedge \left(\sum_{k=1}^{3} f_{2k}\,dx^k\right) \wedge \left(\sum_{l=1}^{3} f_{3l}\,dx^l\right)$$

$$= \sum_{j,k,l=1}^{3} f_{1j} f_{2k} f_{3l}\, dx^j \wedge dx^k \wedge dx^l.$$

证明:

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \det\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} dx^1 \wedge dx^2 \wedge dx^3.$$

3. 设 $\phi \in \Omega_1$. 证明 $d(d\phi) = 0$.