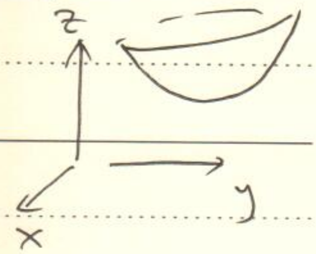
实际上我们可以通过参数变换以及对曲面作合同变换来使公式 $K(u,v)=\dfrac{f_{uu}f_{vv}-f_{uv}^2}{f_u^2+f_v^2+1}$ 有更简洁的形式.



回忆一点处切平面与 x-y 平面的夹角 θ 满足

$$\cos\theta=\langle n,(0,0,1)\rangle$$

$$=\frac{1}{\sqrt{f_u^2+f_v^2+1}}$$

易见,如果一点处的切平面与 x-y 平面平行,则有 $K=f_{uu}f_{vv}-f_{uv}^2$. 此时 $f(u,v)$ 即为点 $r(u,v)$ 到该点处切平面的"带符号"距离(即高度函数).

下面,我们讨论如何严格地实现这一点. 不妨就考虑点 $r(0,0)=(0,0,f(0,0))$ 处的高斯曲率. 在 $r(0,0)$ 处的切平面上我们取一组标准正交基切向量 e_1,e_2. 我们考虑如下的参数曲面(曲) 基于 $r:D\to E^3$ $(u,v)\mapsto r(u,v)=(u,v,f(u,v))$

$$\tilde{r}:D\longrightarrow E^3$$

$$(u,v)\longmapsto \tilde{r}(u,v)=(\bar{u}(u,v),\bar{v}(u,v),h(u,v))$$

这里 $h(u,v):=\langle r(u,v)-r(0,0),n(0,0)\rangle$ 为 $r(u,v)$ 点的高度函数

$$\begin{cases}\bar{u}(u,v)=\langle r(u,v)-r(0,0)-h(u,v)n(0,0),e_1\rangle\\ \bar{v}(u,v)=\langle r(u,v)-r(0,0)-h(u,v)n(0,0),e_2\rangle\end{cases}$$

可见 $\tilde{r}=\tilde{r}(u,v)$ 与 $r=r(u,v)$ 只相差一个合同变换 T. 即

$$\tilde{r}=T\circ r$$

这时 $h(u,v)$ 满足 $h_u(0,0)=\langle r_u(0,0),n(0,0)\rangle=0$

$$h_v(0,0)=\langle r_v(0,0),n(0,0)\rangle=0.$$

但是 $\tilde{r}$ 还不是参数化为 h 的图像了. 为了修正这一点,我们

想，作参数变换

$$\begin{cases}\bar{u} = \bar{u}(u,v) = \langle r(u,v) - r(0,0) - h(u,v)n(0,0), e_1 \rangle \\ \bar{v} = \bar{v}(u,v) = \langle r(u,v) - r(0,0) - h(u,v)n(0,0), e_2 \rangle \end{cases}$$

可以检查

$$J = \begin{pmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{pmatrix} = \begin{pmatrix} \langle r_u - h_u n(0,0), e_1 \rangle & \langle r_u - h_u n(0,0), e_2 \rangle \\ \langle r_v - h_v n(0,0), e_1 \rangle & \langle r_v - h_v n(0,0), e_2 \rangle \end{pmatrix}$$

在 $(0,0)$ 处，$\det J = \det \begin{pmatrix} \langle r_u(0,0), e_1 \rangle & \langle r_u(0,0), e_2 \rangle \\ \langle r_v(0,0), e_1 \rangle & \langle r_v(0,0), e_2 \rangle \end{pmatrix}$

$\neq 0$. （因为 r_u, r_v 线性无关）

故在 $(0,0)$ 的充分小邻域上（仍记为 D），上述参数变换是容许的.

重新参数化后，我们得到

$$\bar{r} : (\bar{u},\bar{v}) \longrightarrow (u,v) \xrightarrow{\tilde{r}} \tilde{r}(u,v)$$

$$\bar{r}(\bar{u},\bar{v}) := \tilde{r}(u(\bar{u},\bar{v}), v(\bar{u},\bar{v}))$$
$$= (\bar{u}, \bar{v}, \underbrace{h(u(\bar{u},\bar{v}), v(\bar{u},\bar{v}))}_{:= \bar{f}(\bar{u},\bar{v})})$$

可以检查 $\bar{f}_{\bar{u}}(0,0) = h_u(0,0)\frac{\partial u}{\partial \bar{u}} + h_v(0,0)\frac{\partial v}{\partial \bar{u}} = 0$

$\bar{f}_{\bar{v}}(0,0) = h_u(0,0)\frac{\partial u}{\partial \bar{v}} + h_v(0,0)\frac{\partial v}{\partial \bar{v}} = 0.$

故而 $\bar{K}(0,0) = \bar{f}_{\bar{u}\bar{u}}\bar{f}_{\bar{v}\bar{v}} - \bar{f}_{\bar{u}\bar{v}}^2$.

前面已经验证高斯曲率不依赖于参数变换。下证

性质 对正则曲面片 $r = r(u,v)$ 和 $\tilde{r} = \tilde{r}(u,v) = T \circ r(u,v)$，$T$ 为合同变换 有 $K(u,v) = \tilde{K}(u,v)$.

证明：易见 $\tilde{r}_u = (T \circ r)_u = T \circ r_u$，$\tilde{r}_v = T \circ r_v$.

因此合同变换保内积，故有 $\tilde E=E$，$\tilde F=F$，$\tilde G=G$. 如果大家知道了高斯绝妙定理，就可得到 $\tilde E,\tilde F,\tilde G$ 保持不变 $\Rightarrow$ 高斯曲率不变。此为后话，我们这里可直接验证.

$$\tilde r_u\wedge\tilde r_v = T\circ r_u\wedge T\circ r_v = (\det T)\,T(r_u\wedge r_v).$$

故而
$$\tilde n = \frac{\tilde r_u\wedge\tilde r_v}{|\tilde r_u\wedge\tilde r_v|} = (\det T)\,T\circ n$$

$$\tilde n_u\wedge\tilde n_v = \underbrace{(\det T)^2}_{=(\pm1)^2}\,T\circ n_u\wedge T\circ n_v = (\det T)\,T(n_u\wedge n_v)$$

故
$$\tilde K(u,v) = \frac{\tilde n_u\wedge\tilde n_v}{\tilde r_u\wedge\tilde r_v} = \frac{(\det T)\,T(n_u\wedge n_v)}{(\det T)\,T(r_u\wedge r_v)} = \frac{n_u\wedge n_v}{r_u\wedge r_v} = K(u,v).$$
□

接下来，我们继续调整
$$\bar r(\bar u,\bar v) = (\bar u,\bar v,\bar f(\bar u,\bar v)).\quad (\bar u,\bar v)\in\bar D.$$

在 x-y 平面上，定义旋转 $R_\theta := \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

我们不妨设 $R_\theta(\bar D)=\bar D$.（取 $\bar D$ 为充分小圆盘即可）.

则有
$$\bar D\xrightarrow{R_\theta}\bar D\xrightarrow{\bar f}\mathbb{R}$$
$$(\tilde u,\tilde v)\longmapsto R_\theta(\tilde u,\tilde v)\longmapsto \bar f\circ R_\theta(\tilde u,\tilde v)$$

则参数曲面
$$\tilde r(\tilde u,\tilde v) := (\tilde u,\tilde v,\underbrace{\bar f\circ R_\theta(\tilde u,\tilde v)}_{:=\tilde f(\tilde u,\tilde v)})$$

仍是一个正则曲面片，它实际上与 $\bar r(\bar u,\bar v)=(\bar u,\bar v,\bar f(\bar u,\bar v))$ 只差一个绕z轴的旋转变换！即 $\tilde r$ 与 $\bar r$ 只相差合同变换，故相应点处的高斯曲率不变.

可以检查 $\tilde f=\bar f\circ R_\theta(\tilde u,\tilde v) = \bar f(\tilde u\cos\theta-\tilde v\sin\theta,\ \tilde u\sin\theta+\tilde v\cos\theta)$

$$\tilde f_{\tilde u}(0,0) = \bar f_{\bar u}\cos\theta+\bar f_{\bar v}\sin\theta\Big|_{(0,0)} = 0$$

$$\tilde f_{\tilde v}(0,0) = \bar f_{\bar u}(-\sin\theta) + \bar f_{\bar v}\cos\theta\Big|_{(0,0)} = 0.$$

$$\tilde f_{\tilde u\tilde v} = \bar f_{\bar u\bar u}(-\sin\theta\cos\theta) + \bar f_{\bar u\bar v}(\cos^2\theta - \sin^2\theta) + \bar f_{\bar v\bar v}\sin\theta\cos\theta$$

$$= \cos 2\theta\, \bar f_{\bar u\bar v} + \frac{\sin 2\theta}{2}\left(\bar f_{\bar v\bar v} - \bar f_{\bar u\bar u}\right)$$

在(0,0)处，如 $\bar f_{\bar v\bar v}(0,0) \neq \bar f_{\bar u\bar u}(0,0)$，则取 θ_0 s.t.

$$\tan 2\theta_0 = -\frac{2\bar f_{\bar u\bar v}(0,0)}{\bar f_{\bar v\bar v}(0,0) - \bar f_{\bar u\bar u}(0,0)}$$

如 $\bar f_{\bar v\bar v}(0,0) = \bar f_{\bar u\bar u}(0,0)$，则取 $\theta_0 = \frac{\pi}{4}$ s.t. $\cos 2\theta_0 = 0$.

这时总有 $\tilde f_{\tilde u\tilde v}(0,0) = 0$.

总结起来，对 $\tilde r(\tilde u,\tilde v) = (\tilde u, \tilde v, \tilde f(\tilde u,\tilde v))$ 满足

$$\tilde f_{\tilde u}(0,0) = \tilde f_{\tilde v}(0,0) = 0$$

$$\tilde f_{\tilde u\tilde v}(0,0) = 0.$$

故 $\tilde K(0,0) = \tilde f_{\tilde u\tilde u}(0,0)\cdot\tilde f_{\tilde v\tilde v}(0,0)$. !!