HOMEWORK 1: DIFFERENTIAL MANIFOLDS AND RIEMANNIAN METRICS

RIEMANNIAN GEOMETRY, SPRING 2020

1. (Tangent bundles)

Let M be an n dimensional manifold and let $TM = \{(p, v) : p \in M, v \in T_pM\}$. Let $\{U_\alpha, x_\alpha\}_{\alpha\in A}$ be an atlas of M. For any $\alpha \in A$, denote by

$$X_\alpha := \{(p, v) : p \in U_\alpha, v \in T_pM\}$$

a subset of TM and assign a topology τ_α to it such that the following map is a homeomorphism:

$$\begin{aligned}\phi_\alpha :\ & x_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \longrightarrow X_\alpha, \\ & (x_\alpha(p), (v^1, \ldots, v^n)) \mapsto (p, v^i \frac{\partial}{\partial x^i_\alpha}).\end{aligned}$$

(i) Show that there exists a topology τ on TM such that τ induces upon each X_α the topology τ_α and TM together with the topology τ is a $2n$ dimensional manifold.

(ii) Suppose further that M is a differentiable manifold with $\{U_\alpha, x_\alpha\}$ being a differentiable atlas. Show that TM (with the above topology τ) admits a differentiable structure.

Hint: You are allowed to use the following theorem about gluing topological spaces:

Theorem 0.1. *Let X be a set. Let $\{X_i\}$ be a collection of subsets whose union is X. Suppose on each X_i, there is a topology τ_i, and that τ_i's are compatible in the following sense: $X_i \cap X_j$ is open in each X_i and X_j, and the induced topologies on $X_i \cap X_j$ from both X_i and X_j coincide. Then there exists a unique topology on X that induces upon each X_i the topology τ_i.*

2. (Riemannian measure)

Let (M, g) be an n dimensional Riemannian manifold. Recall that we have defined the following positive linear functional Λ on $C^0_0(M)$:

$$\Lambda f := \sum_\alpha \int_{x_\alpha(U_\alpha)} \phi_\alpha \circ x_\alpha^{-1} \cdot f \circ x_\alpha^{-1} \sqrt{\det(g^{x_\alpha}_{ij})}\, dx^1_\alpha \cdots dx^n_\alpha,$$

where $\{U_\alpha, x_\alpha\}$ is an locally finite atlas and $\{\phi_\alpha\}$ is a *partition of unity* subordinate to it, $g^{x_\alpha}_{ij} = g(\frac{\partial}{\partial x^i_\alpha}, \frac{\partial}{\partial x^j_\alpha})$.

We define a nonnegative function μ on the set of all subsets of M as below: Define for every open set $U \subset M$

$$\mu(U) := \sup\left\{\Lambda f :\ f \in C^0_0(M), 0 \le f \le 1, \mathrm{supp}(f) \subset U\right\},$$

and then define for any subset $E \subset M$

$$\mu(E) := \inf\{\mu(U) :\ E \subset U, U \text{ is open}\}.$$

Consider the following particular class of subsets as a candidate for a σ-algebra:

$$\mathfrak{M} := \{E \subset M : E \cap K \in \mathfrak{M}_F \text{ for any compact subset } K\},$$

where

$$\mathfrak{M}_F := \{E \subset M : \mu(E) < \infty, \mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}\}.$$

Prove that $\mathfrak{M}$ is indeed a σ-algebra and contains all Borel sets in M and μ is a regular measure on $\mathfrak{M}$.

Hint: In fact, you are asked here to prove the *Riesz Representation Theorem* on a locally compact, σ-compact, Hausdorff topological space. You can read, for example, the Chapter Two of Rudin's book "*Real and Complex analysis*" for a proof.

3. (Spheres)

The **sphere**

$$S^n := \left\{(x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1\right\}$$

is a manifold with the following atlas $\{U_\alpha, y_\alpha\}_{\alpha \in \{1,2\}}$:

$$y_1 : \ U_1 := S^n \setminus \{(0, \dots, 0, 1)\} \longrightarrow \mathbb{R}^n,$$
$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y_1^1, \dots, y_1^n) := \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}}\right).$$

and

$$y_2 : \ U_2 := S^n \setminus \{(0, \dots, 0, -1)\} \longrightarrow \mathbb{R}^n,$$
$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y_2^1, \dots, y_2^n) := \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}}\right).$$

(i) Prove that the above atlas $\{U_\alpha, y_\alpha\}_{\alpha \in \{1,2\}}$ is differentiable.

(ii) Let g be the induced metric of S^n from the standard Euclidean metric of $\mathbb{R}^{n+1}$. Prove that in each chart U_α, the metric matrix $\left(g_{ij}^{y_\alpha}\right)$ is given by

$$g_{ij}^{y_\alpha} = \frac{4}{(1 + \sum_{i=1}^n (y_\alpha^i)^2)^2} \delta_{ij}.$$

(iii) Let μ be the Riemannian measure of (S^n, g) defined in Problem 2. Compute $\mu(S^n)$.

4. (Hyperbolic spaces)

The **hyperboloid** is

$$H^n := \left\{(x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} (x^i)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0\right\}.$$

Consider the following map

$$y : \ H^n \longrightarrow B_1(0) := \left\{(y^1, \dots, y^n) \in \mathbb{R}^n : \sum_{i=1}^{n} (y^i)^2 < 1\right\} \subset \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y^1, \dots, y^n) := \left(\frac{x^1}{1 + x^{n+1}}, \dots, \frac{x^n}{1 + x^{n+1}}\right).$$

(i) Prove that the above map y is a diffeomorphism between H^n and $B_1(0)$. Therefore, $\{H^n, y\}$ is a differentiable atlas of H^n.

(ii) Let g be the Riemannian metric of H^n induced from $\mathbb{R}^{n+1}$ assigned with the *Lorentz metric*:

$$g_L = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n - dx^{n+1} \otimes dx^{n+1}.$$

Prove that in the global chart $\{H^n, y\}$, the metric matrix (g_{ij}) is given by

$$g_{ij} = \frac{4}{(1 - \sum_{i=1}^n (y^i)^2)^2}\delta_{ij}.$$

(iii) Let μ be the Riemannian measure of (H^n, g) defined in Problem 2. Compute $\mu(H^n)$.

Remark: *Lorentz manifolds* are the spaces occurring in general relativity. For example, the above mentioned $(\mathbb{R}^{n+1}, g_L)$ is a special Lorentz manifold, which is often referred to as a *Minkowski space*. A tangent vector v of a Lorentz manifold can have *negative, positive, or vanishing* norm $\|v\| := \sqrt{g_L(v, v)}$, which is called a *time-like, space-like, or light-like* tangent vector, respectively. Submanifolds of Lorentz manifolds whose tangent vectors are all space-like are Riemannian manifolds with respect to the induced metric. The hyperboloid H^n assigned with the induced metric g, which is often referred to as a *hyperbolic space*, is such an example.

5. (Flat tori)

Let $w_1, w_2, \ldots, w_n \in \mathbb{R}^n$ be linearly independent. We consider $z_1, z_2 \in \mathbb{R}^n$ as equivalent if there exist integers $m_1, m_2, \ldots, m_n$ such that

$$z_1 - z_2 = \sum_{i=1}^n m_i w_i.$$

Let π be the projection mapping $z \in \mathbb{R}^n$ to its equivalence class.

The **torus**

$$T^n := \pi(\mathbb{R}^n)$$

is a manifold with the following atlas $\{U_\alpha, z_\alpha\}$:

$$U_\alpha := \pi(\Delta_\alpha),$$
$$z_\alpha := \left(\pi_{|\Delta_\alpha}\right)^{-1} : U_\alpha \longrightarrow \Delta_\alpha \subset \mathbb{R}^n,$$

where Δ_α is any open subset of $\mathbb{R}^n$ which does not contain any pair of equivalent points.

(i) Prove that the above atlas $\{U_\alpha, z_\alpha\}$ is differentiable.

(ii) Prove that there exists a Riemannian metric g on T^n, such that the map z_α is an *isometry* between the Riemannian manifolds $(U_\alpha, g_{|U_\alpha})$ and $(\Delta_\alpha, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the standard Euclidean metric on $\Delta_\alpha \subset \mathbb{R}^n$.

(iii) Let μ be the Riemannian measure of (T^n, g) defined in Problem 2. Compute $\mu(T^n)$.

Remark: A differentiable map $h : M \to N$ is a *local isometry* between Riemannian manifolds if for every $p \in M$ there exists a neighborhood U of p for which $h_{|U} : U \to h(U)$ is an isometry and $h(U)$ is open in N. In terms of this terminology, you are asked to show in (ii) that there exists a Riemannian metric g on T^n such that the projection $\pi : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \longrightarrow (T^n, g)$ is a local isometry. A torus assigned with such a metric is called a *flat torus*.