HOMEWORK 2: GEODESICS

RIEMANNIAN GEOMETRY, SPRING 2020

1. (Christoffel symbols)

Let $(U, x = (x^1, \ldots, x^n))$ be a chart of a Riemannian manifold M. Let

$$(x^1, \ldots, x^n) \to (y^1, \ldots, y^n)$$

be a smooth coordinate change, and the Riemannian metric can be written as $g_{ij}(x)dx^i \otimes dx^j$ and $h_{\alpha\beta}(y)dy^\alpha \otimes dy^\beta$ respectively.

(i) Show the transformation formula of g^{ij} under the coordinate change is

$$g^{ij}(x) = h^{\alpha\beta}(y(x)) \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}.$$

(ii) Compute the transformation formulae of the Christoffel symbols Γ^i_{jk} under the coordinate change. Do they define a tensor?

(iii) Let $\gamma : [a, b] \to U$ be a smooth curve. Denote $\dot{x}^i(t) := \frac{d}{dt} x^i(\gamma(t))$. Compute the transformation formula of

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)$$

under the coordinate change.

Remark: *Elwin Christoffel* (1829-1900) was noted for his work in mathematical analysis, in which he was a follower of *Dirichlet* and *Riemann*. He wrote important papers which contributed to the development of the tensor calculus of *Gregorio Ricci-Curbastro* and *Tullio Levi-Civita*. The *Christoffel symbols* which he introduced are fundamental in the study of tensor analysis. The *Christoffel reduction theorem*, so named by *Klein*, solves the *local equivalence problem for two quadratic differential forms*. *Paul Butzer* once commented:

> *The procedure Christoffel employed in his solution of the equivalence problem is what Gregorio Ricci-Curbastro later called* **covariant differentiation**, *Christoffel also used the latter concept to define the basic* **Riemann-Christoffel curvature tensor**. ... *The importance of this approach and the two concepts Christoffel introduced, at least implicitly, can only be judged when one considers the influence it has had.*

Indeed this influece is clearly seen since this allowed *Ricci-Curbastro* and *Levi-Civita* to develop a coordinate free differential calculus which *Einstein*, with the help of *Grossmann*, turned into the tensor analysis mathematical foundation of general relativity.
(*Read more at* http://mathshistory.st-andrews.ac.uk/Biographies/Christoffel.html)

2. Geodesic equation in Finsler geometry

Let M be an n-dimensional smooth manifold. Let $TM = \{(x, y) : x \in M, y \in T_xM\}$ be the tangent bundle of M.

A *Finsler structure* of M is a function

$$F : TM \to [0, \infty)$$

with the following properties:

(1) *Regularity*: F is C^∞ on $TM \setminus 0$.
(2) *Absolute homogeneity*: $F(x, \lambda y) = |\lambda| F(x, y)$ for all $\lambda \in \mathbb{R}$.
(3) *Strong convexity*: The $n \times n$ Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at every point of $TM \setminus 0$. (Explanation of y^i: For any basis $\{\frac{\partial}{\partial x^i}\}$, express y as $y^i \frac{\partial}{\partial x^i}$. The Finsler structure F is then a function of $(x^1, \dots, x^n, y^1, \dots, y^n)$, and

$$\left[\frac{1}{2} F^2 \right]_{y^i y^j} := \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{1}{2} F^2 \right].$$

It can be checked that the positive-definiteness is independent of the choice of $\{\frac{\partial}{\partial x^i}\}$.)

Let $\gamma : [a, b] \to M$ be a smooth curve in M. Suppose the parametrization of γ is regular, i.e., $\dot\gamma(t) \neq 0$, $\forall t \in [a, b]$. We can define the length and energy of γ to be

$$L(\gamma) := \int_a^b F(\gamma(t), \dot\gamma(t)) dt,$$
$$E(\gamma) := \frac{1}{2} \int_a^b F^2(\gamma(t), \dot\gamma(t)) dt,$$

respectively.

(i) Prove that $L(\gamma)$ does not depend on the choice of a regular parametrization.

(ii) Prove that $L(\gamma)^2 \leq 2(b - a)E(\gamma)$, and characterize the case when "=" holds.

(iii)Suppose that the image $\gamma([a, b])$ falls in a chart $(U, x = (x^1, \dots, x^n))$. Denote by

$$\gamma(t) := (x^1(t), \dots, x^n(t)).$$

Show that the Euler-Lagrange equation for $E(\gamma)$ (defined to be the geodesic equation) is

$$\ddot x^\ell + \frac{1}{2} g^{i\ell} \left(\left[F^2\right]_{x^j y^i} y^j - \left[F^2\right]_{x^i} \right) = 0, \ \forall \ell = 1, \dots, n,$$

where $\left(g^{i\ell}\right)$ is the inverse matrix of (g_{ij}).

Remark: The concept of Riemannian metric was introduced by Riemann in his habilitation address, entitled *Über die Hypothesen, welche der Geometrie zugrunde liegen.* Riemannian metrics are induced by Euclidean scalar products on the tangent spaces. In fact, Riemann also suggested to consider more general metrics obtained by taking metrics on the tangent spaces that are not induced by a scalar product. Such metrics were first systematically investigated by Finsler and are therefore called *Finsler metric.*

3. (Spheres)

Recall that the **sphere**

$$S^n := \left\{ (x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}$$

is a differentiable manifold with the following differentiable atlas $\{U_\alpha, y_\alpha\}_{\alpha\in\{1,2\}}$:

$$y_1:\ U_1 := S^n \setminus \{(0,\dots,0,1)\} \longrightarrow \mathbb{R}^n,$$

$$(x^1,\dots,x^n,x^{n+1}) \mapsto (y_1^1,\dots,y_1^n) := \left(\frac{x^1}{1-x^{n+1}},\dots,\frac{x^n}{1-x^{n+1}}\right).$$

and

$$y_2:\ U_2 := S^n \setminus \{(0,\dots,0,-1)\} \longrightarrow \mathbb{R}^n,$$

$$(x^1,\dots,x^n,x^{n+1}) \mapsto (y_2^1,\dots,y_2^n) := \left(\frac{x^1}{1+x^{n+1}},\dots,\frac{x^n}{1+x^{n+1}}\right).$$

Recall that the induced metric g of S^n from the standard Euclidean metric of $\mathbb{R}^{n+1}$ is given in local coordinates by

$$g_{ij}^{y_\alpha} = \frac{4}{(1+\sum_{i=1}^n (y_\alpha^i)^2)^2}\delta_{ij}.$$

(i) Compute the Christoffel symbols in the chart (U_1, y_1).

(ii) Write down the system of differential equations satisfied by the geodesics in the chart (U_1, y_1) and determine their solutions.

(iii) Determine the the injective radius and the cut locus of the north pole $(0,\dots,0,1)$. Is U_1 a normal neighborhood of the north pole $(0,\dots,0,1)$? Is U_1 a **totally** normal neighborhood of the north pole $(0,\dots,0,1)$?

4. (Hyperbolic spaces)

Recall that the **hyperboloid**

$$H^n := \left\{(x^1,\dots,x^n,x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n (x^i)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0\right\}$$

is a differentiable manifold with the following chart:

$$y:\ H^n \longrightarrow B_1(0) := \left\{(y^1,\dots,y^n) \in \mathbb{R}^n : \sum_{i=1}^n (y^i)^2 < 1\right\} \subset \mathbb{R}^n,$$

$$(x^1,\dots,x^n,x^{n+1}) \mapsto (y^1,\dots,y^n) := \left(\frac{x^1}{1+x^{n+1}},\dots,\frac{x^n}{1+x^{n+1}}\right).$$

Let g be the Riemannian metric of H^n given by

$$g_{ij} = \frac{4}{(1-\sum_{i=1}^n (y^i)^2)^2}\delta_{ij}.$$

(i) Compute the Christoffel symbols and write down the system of differential equations satisfied by the geodesics.

(ii) Determine the geodesics of H^n through the point $(0,\dots,0,1) \in \mathbb{R}^{n+1}$ (whose coordinate is $(0,\dots,0) \in B_1(0) \subset \mathbb{R}^n$).

(iii) Is H^n a complete Riemannian manifold?

Hint: We point out the following useful fact: The function

$$y(t) := \frac{e^t - 1}{e^t + 1},\ t \in [0,\infty)$$

is a solution of the following ODE:

$$\begin{cases} \ddot{y}(t) + \frac{2y(t)}{1-y(t)^2}\dot{y}(t)^2 = 0, \\ y(0) = 0. \end{cases}$$

5. (Flat tori)

Let $w_1, w_2, \dots, w_n \in \mathbb{R}^n$ be linearly independent. We consider $z_1, z_2 \in \mathbb{R}^n$ as equivalent if there exist integers $m_1, m_2, \dots, m_n$ such that

$$z_1 - z_2 = \sum_{i=1}^{n} m_i w_i.$$

Let π be the projection mapping $z \in \mathbb{R}^n$ to its equivalence class.

Recall that the **torus**

$$T^n := \pi(\mathbb{R}^n)$$

is a differentiable manifold with the following differentiable atlas $\{U_\alpha, z_\alpha\}$:

$$U_\alpha := \pi(\Delta_\alpha),$$
$$z_\alpha := \left(\pi_{|\Delta_\alpha}\right)^{-1} : U_\alpha \longrightarrow \Delta_\alpha \subset \mathbb{R}^n,$$

where Δ_α is any open subset of $\mathbb{R}^n$ which does not contain any pair of equivalent points.

Let g be the Riemannian metric on T^n, such that the map z_α is an *isometry* between the Riemannian manifolds $(U_\alpha, g_{|U_\alpha})$ and $(\Delta_\alpha, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the standard Euclidean metric on $\Delta_\alpha \subset \mathbb{R}^n$. Note that in this case

$$\pi : \mathbb{R}^n \to T^n$$

is a local isometry.

Let us consider the standard torus T_0^n where we take $w_i = (0, \dots, 0, 1, 0, \dots, 0)$ to be the i-th unit vector in $\mathbb{R}^n$.

(i) Determine the injective radius and cut locus of any point in T_0^n.

(ii) Is T_0^n a complete Riemannian manifold?

6. (Completeness)

(i) Assume that (M, g) has the property that all normal geodesics exist for a fixed time $\epsilon > 0$. Show that (M, g) is geodesically complete.

(ii) Let (M, g) be a metrically complete Riemannian manifold and $\tilde{g}$ is another metric on M such that $\tilde{g} \geq g$. Show that $(M, \tilde{g})$ is also metrically complete.

(iii) Let (M, g) be a Riemannian manifold which admits a proper Lipschitz function $f : M \to \mathbb{R}$. Show that (M, g) is complete. (Recall that a function between topological spaces is called proper if inverse images of compact subsets are compact.)

Remark: The Hopf-Rinow Theorem is named after German mathematicians *Heinz Hopf* (1894-1971) and his student *Willi Rinow* (1907-1979), who published it in *Über den Begriff der vollständigen differentialgeometrischen Fläche, Commentarii Mathematici Helvetici, **3**, 209-225 (1931)* (The title in English: *On the concept of complete differentiable surfaces*). The Hopf-Rinow theorem is generalized to length-metric spaces in the following way: If a length-metric space (M, d) is *complete* and *locally compact* then any two points in M can be connected by a shortest geodesic, and any bounded closed set in M is compact. The theorem does **not** hold in infinite dimensions: *C. J. Atkin (The Hopf-Rinow Theorem is false in infinite dimensions, Bulletin of the London Mathematical Society, **7**(3), 261-266, 1975)* showed that two points in an infinite dimensional complete Hilbert manifold need not be connected by a geodesic (even if you do not require this geodesic to be a shortest curve).

Heinz Hopf worked on the fields of topology and geometry. In his dissertation, *Über Zusammenhänge zwischen Topologie und Metrik von Mannigfaltigkeiten* in 1925 (in English, *connections between topology and metric of manifolds*), he proved that any simply connected *complete* Riemannian 3-manifold of constant sectional curvature is globally isometric to *Euclidean, spherical, or hyperbolic space*. He also studied the indices of zeros of vector fields on hypersurfaces, and connected their sum to curvature. Some six months later he gave a new proof that the sum of the indices of the zeros of a vector field on a manifold is independent of the choice of vector field and equal to the *Euler characteristic* of the manifold. This theorem is now called the *Poincaré-Hopf theorem*. Hopf spent the academic year 1927/28 at Princeton University and at this time he discovered the *Hopf invariant* of maps $S^3 \to S^2$ and proved that the *Hopf fibration* has invariant 1.