HOMEWORK 3: FUNCTION SPACES AND LAPLACE-BELTRAMI OPERATOR

RIEMANNIAN GEOMETRY, SPRING 2020

1. Lipschitz functions and Sobolev spaces

Let (M,g) be a complete Riemmanian manifold. A function $f: M \to \mathbb{R}$ is said to be Lipschitz if there exists $C > 0$ such that for any $x, y \in M$,

$$|u(y) - u(x)| \leq Cd(x,y),$$

where $d(\cdot,\cdot)$ is the distance function associate to g.

(i) Show that a smooth function $f: M \to \mathbb{R}$ is Lipschitz if the norm of its gradient $|\nabla f|$ is bounded.

(ii) Recall the following fact: If Ω is a bounded open subset of $\mathbb{R}^n$ and if $f: \Omega \to \mathbb{R}$ is Lipschitz, then $f \in H^{1,2}(\Omega)$. Here $H^{1,2}(\Omega)$ is the Sobolev space with respect to the Euclidean metric. Using this fact to show that any Lipschitz function $f: M \to \mathbb{R}$ with compact support belongs to $H^{1,2}(\mathrm{vol}_g)$.

2. Energy, Rayleigh quotient, and min-max principle

Let (M,g) be a compact Riemannian manifold and Δ be the Laplace-Beltrami operator. Let all the eigenvalues of Δ be

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_k(g) \leq \cdots.$$

For each $i = 0, 1, 2, \ldots$, denote by v_i the eigenfunction corresponding to $\lambda_i(g)$ with $(v_i, v_i) = 1$, where $(\cdot,\cdot)$ is the L^2-inner product.

The *energy* of a function $f \in H^{1,2}(\mathrm{vol}_g)$ on (M,g) is defined as

$$E_g(f) := \frac{1}{2}\int_M |\nabla f|^2 d\mathrm{vol}_g,$$

and the *Rayleigh quotient* of a function $f \in H^{1,2}(\mathrm{vol}_g)$, $f \neq 0$ on (M,g) is defined as

$$R_g(f) := \frac{\int_M |\nabla f|^2 d\mathrm{vol}_g}{\int_M f^2 d\mathrm{vol}_g}.$$

We observe that

$$R_g(v_i) = \lambda_i(g),\ \forall i = 0, 1, 2, \ldots.$$

(i) Show that for any $f \in H^{1,2}(\mathrm{vol}_g)$ we have

$$2E_g(f) = \sum_{i=1}^{\infty} \lambda_i(g)(f, v_i).$$

(ii) For $k = 0, 1, 2, \ldots$, let

$$V_k := \mathrm{span}\{v_0, v_1, \ldots, v_k\} \subset H^{1,2}(\mathrm{vol}_g)$$

be the $k+1$-dimensional subspace of $H^{1,2}(\mathrm{vol}_g)$ spanned by $v_0, v_1, \ldots, v_k$; Let

$$V_{k-1}^{\perp} := \mathrm{span}\{v_k, v_{k+1}, \ldots\} \subset H^{1,2}(\mathrm{vol}_g)$$

be the subspace vertical to V_{k-1}. Here, we use the notation $V_{-1}^{\perp} = H^{1,2}(\mathrm{vol}_g)$.

Show that

$$\lambda_k(g) = \max_{f \in V_k, f \neq 0} R_g(f) = \min_{f \in V_{k-1}^{\perp}, f \neq 0} R_g(f).$$

(iii) Show that

$$\lambda_k(g) = \min_{\substack{E \subset H^{1,2}(\mathrm{vol}_g) \\ \dim(E) = k+1}} \max_{f \in E, f \neq 0} R_g(f),$$

and

$$\lambda_k(g) = \max_{\substack{E \subset H^{1,2}(\mathrm{vol}_g) \\ \dim(E) = k}} \min_{f \in E^{\perp}, f \neq 0} R_g(f),$$

(iv) Let g and g_0 be two Riemannian metrics on M such that

$$a^2 g_0 \leq g \leq b^2 g_0$$

for some $0 < a^2 < b^2$. For any $f \in H^{1,2}(\mathrm{vol}_g)$, $f \neq 0$, show that

$$\frac{a^n}{b^{n+2}} R_{g_0}(f) \leq R_g(f) \leq \frac{b^n}{a^{n+2}} R_{g_0}(f).$$

For any $k = 0, 1, 2, \ldots$, show that

$$\frac{a^n}{b^{n+2}} \lambda_k(g_0) \leq \lambda_k(g) \leq \frac{b^n}{a^{n+2}} \lambda_k(g_0).$$

Remark: The property claimed in (iv) above tells that if a sequence g_n of Riemannian metrics on M converges to g_0 with respect to C^0 topology, then $\lambda_k(g_n)$ converges to $\lambda_k(g_0)$ for any k. This fact illustrate *the strength of the min-max principle*: indeed, if g_n converges to g_0 with respect to C^0 topology, it is not always true that Δ_{g_n} converges to Δ_{g_0}.