HOMEWORK 4: CONNECTIONS, PARALLELISM, AND COVARIANT DERIVATIVES

RIEMANNIAN GEOMETRY, SPRING 2020

1. Torsion tensor

Let M be a smooth manifold. Let ∇ and $\overline{\nabla}$ be two affine connections on M. We define

$$D(X,Y) := \nabla_X Y - \overline{\nabla}_X Y, \ \forall \ X,Y \in \Gamma(TM).$$

(i) Prove that D is a tensor, that is, D is linear over C^∞ functions in both arguments.

(ii) Prove that there is a unique way to write

$$D = S + A$$

with S *symmetric* and A *alternating*, i.e., $S(X,Y) = S(Y,X)$ and $A(X,Y) = -A(Y,X)$.

(iii) Prove that ∇ and $\overline{\nabla}$ have that same torsion if and only if $A = 0$.

(iv) A parametrized curve $\gamma = \gamma(t)$ on M is called a geodesic with respect to an affine connection ∇ if $\left(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\right)(\gamma(t)) = 0$ for any t. Prove that the following are equivalent:

(a) ∇ and $\overline{\nabla}$ have the same geodesics;
(b) $D(X,X) = 0$ for any $X \in \Gamma(TM)$.
(c) $S = 0$.

(v) Prove that if ∇ and $\overline{\nabla}$ have the same geodesics and the same torsion, then $\nabla = \overline{\nabla}$.

(vi) Prove that for any affine connection ∇ on M, there exists a unique affine connection $\overline{\nabla}$ with the same geodesics and with torsion 0. (*Hint*: Consider the connection $\overline{\nabla}_X Y := \nabla_X Y - \frac{1}{2}T(X,Y)$, where T is the torsion of ∇.)

Remark: We can define two affine connections with the same geodesics to be equivalent. Then all affine connections on M can be divided into equivalent classes. The above discussions tell us that each equivalent class has *exactly one* connection with zero torsion.

2. Connections on spheres

Let S^n be the sphere with the induced metric g from the Euclidean metric in $\mathbb{R}^{n+1}$. We denote by $\overline{\nabla}$ the canonical Levi-Civita connection on $\mathbb{R}^{n+1}$. For any $X, Y \in \Gamma(T\mathbb{S}^n)$, one can extend X, Y to smooth vector field $\overline{X}, \overline{Y}$ on $\mathbb{R}^{n+1}$, at least near $\mathbb{S}^n$.

By locality, the vector $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any $p \in \mathbb{S}^n$ depends only on $\overline{X}(p) = X(p)$ and the vectors $\overline{Y}(q) = Y(q)$ for $q \in \mathbb{S}^n$. That is, $\overline{\nabla}_{\overline{X}}\overline{Y}$ is independent of the extension of X, Y we choose. So we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\overline{X}}\overline{Y}$ at points on $\mathbb{S}^n$.

We define $\nabla_X Y$ to be the orthogonal projection of $\overline{\nabla}_X Y$ onto the tangent space of $\mathbb{S}^n$, i.e.,

$$\nabla_X Y := \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathbf{n}\rangle \mathbf{n},$$

where $\mathbf{n}$ is the unit out normal vector on $\mathbb{S}^n$.

(i) Prove that ∇ is an affine connection on $\mathbb{S}^n$.

(ii) Prove that ∇ is the Levi-Civita connection of $(\mathbb{S}^n, g)$.

Remark: This is in fact a general way to construct Levi-Civita connections on a Riemannian manifold. Recall that any Riemannian manifold (M,g) can be embedded isometrically to a Euclidean space E of high enough dimension. For any $p \in M$, we have the orthogonal projection map

$$\pi(p) : T_pE \to T_pM.$$

The composition of this orthogonal projection map with the canonical Levi-Civita connection on E (given by directional derivatives) produces the Levi-Civita connection on (M, g).

3. Covariant derivatives of tensor fields via parallel transport

Recall that for an isomorphism $\varphi : V \to W$ between two vector spaces V and W, there is an adjoint isomorphism

$$\varphi^* : W^* \to V^*,$$

between their dual spaces. For $\alpha \in W^*$, we have

$$\varphi(\alpha)(v) := \alpha(\varphi(v)), \ \forall \ v \in V.$$

Then, for any $v_i \in V, \ \alpha^j \in V^*$, we define

$$\widetilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s) = \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s).$$

By linearity, we can extend $\widetilde{\varphi}$ to be defined on all (r, s)-tensor, $\otimes^{r,s}V$, over V. This defines an isomorphism

$$\widetilde{\varphi} : \otimes^{r,s}V \to \otimes^{r,s}W.$$

Let M be a smooth manifold with an affine connection ∇. Let $c : I \to M$ be a smooth curve in M with $c(0) = p \in M$ and $\dot{c}(0) = X_p \in T_pM$. Recall that the parallel transport

$$P_{c,t} : T_{c(0)}M \to T_{c(t)}M,$$

is an isomorphism. As described above, we can extend it to be an isomorphism

$$\widetilde{P}_{c,t} : \otimes^{r,s}T_{c(0)}M \to \otimes^{r,s}T_{c(t)}M.$$

For any $A \in \Gamma(\otimes^{r,s}TM)$, we define

$$\nabla_{X_p}A := \lim_{h\to 0} \frac{1}{h} \left(\widetilde{P}_{c,h}^{-1} A(c(h)) - A(p) \right).$$

Let $Y \in \Gamma(TM), w, \eta \in \Gamma(T^*M)$. Consider the $(1, 2)$-tensor filed $K := Y \otimes w \otimes \eta$.

(i) Show that

$$\nabla_{X_p}K = \nabla_{X_p}Y \otimes w \otimes \eta + Y \otimes \nabla_{X_p}w \otimes \eta + Y \otimes w \otimes \nabla_{X_p}\eta.$$

(ii) Let $C : \Gamma(\otimes^{1,2}TM) \to \Gamma(\otimes^{0,1}TM)$ be the contraction map that pairs the first vector with the first covector. For example, $CK = w(Y)\eta$. Show that

$$\nabla_{X_p}(CK) = C(\nabla_{X_p}K).$$

4. Induced connections

Let M, N be two smooth manifold and $\varphi : N \to M$ be a smooth map. A vector field along φ is an assignment

$$x \in N \mapsto T_{\varphi(x)}M.$$

Let $\{E_i\}_{i=1}^n$ be a frame field in a chart U of $\varphi(x) \in M$. Then for any $x \in \varphi^{-1}(U)$, we have

$$V(x) = V^i(x) E_i(\varphi(x)).$$

Let $u \in T_x N$. We define

$$\widetilde{\nabla}_u V := u(V^i)(x) E_i(\varphi(x)) + V^i(x) \nabla_{d\varphi(u)} E_i, \tag{0.1}$$

where ∇ is an affine connection on M.

(i) Check that $\widetilde{\nabla}_u V$ is well defined, i.e., (0.1) is independent of the choices of chart U and $\{E_i\}$.

(ii) Let g be a Riemannian metric on M. Prove that if ∇ on M is compatible with g, then for vector fields V, W along φ, and $u \in T_x N$, we have

$$u\langle V, W \rangle = \langle \widetilde{\nabla}_u V, W \rangle + \langle V, \widetilde{\nabla}_u W \rangle.$$

(iii) Prove that if ∇ on M is torsion free, then for any $X, Y \in \Gamma(TN)$, we have

$$\widetilde{\nabla}_X d\varphi(Y) - \widetilde{\nabla}_Y d\varphi(X) - d\varphi([X, y]) = 0.$$

5. First variation formula for piecewise smooth curves

Let $c : [0, a] \to M$ be a piecewise smooth curve. That is, there exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = a$$

such that c is smooth on each interval $[t_i, t_{i+1}]$.

(i) At the break points t_i, there are two possible values for the velocity vector filed along c: a right derivative and a left derivative:

$$\dot{c}(t_i^+) = \frac{dc}{dt}_{|[t_i, t_{i+1}]}(t_i), \ \dot{c}(t_i^-) = \frac{dc}{dt}_{|[t_{i-1}, t_i]}(t_i).$$

Let $F : [0, a] \times (-\epsilon, \epsilon) \to M$ be a piecewise smooth variation of c, that is, F is smooth on each $[t_i, t_{i+1}] \times (-\epsilon, \epsilon)$ and $\frac{\partial F}{\partial s}$ is well defined even at t_i's. Derive the *First Variation Formula* of the energy functional.

(ii) Let $V(t)$ be a piecewise smooth vector filed along the curve c. Show that there exists a variation $F : [0, a] \times (-\epsilon, \epsilon) \to M$ such that $V(t)$ is the variational field of F; in addition, If $V(0) = V(a) = 0$, it is possible to choose F as a proper variation. (Hint: Use exponential maps.)

(iii) (Characterization of geodesics) Prove that a piecewise smooth curve $c : [0, a] \to M$ is a geodesic if and only if, for every proper variation F of c, we have

$$E'(0) = 0.$$

6. A natural extension of Gauss' lemma

Let N_1, N_2 be two submanifolds of a complete Riemannian manifold (M, g), and let $\gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) \in N_1$, $\gamma(a) \in N_2$ and γ is the shortest curve from N_1 to N_2. Prove that $\dot{\gamma}(0)$ is perpendicular to $T_{\gamma(0)} N_1$, and $\dot{\gamma}(a)$ is perpendicular to $T_{\gamma(t)} N_2$.