HOMEWORK 5: CURVATURE

RIEMANNIAN GEOMETRY, SPRING 2020

1. (Riemmannian curvature tensor)

Prove that for any $X, Y, Z, W \in \Gamma(TM)$, we have

$$-6\,\langle R(X,Y)Z,W\rangle = \frac{\partial^2}{\partial s \partial t}_{|s=t=0} [\,\langle R(X+sZ,Y+tW)(Y+tW),X+sZ\rangle \\ -\langle R(X+sW,Y+tZ)(Y+tZ),X+sW\rangle]$$

(Hint: using the first Bianchi identity.)

Remark: This gives an alternative proof of the following fact: The values

$$\langle R(X,Y)Z,W\rangle\,,\ \ \forall X,Y,Z,W\in\Gamma(TM)$$

are determined by the values

$$\langle R(X,Y)Y,X\rangle\,,\ \ \forall X,Y\in\Gamma(TM).$$

2. (Spheres)

Recall that the **sphere**

$$S^n := \left\{(x^1,\ldots,x^n,x^{n+1})\in\mathbb{R}^{n+1} : \sum_{i=1}^{n+1}(x^i)^2 = 1\right\}$$

is a differentiable manifold with the following differentiable atlas $\{U_\alpha, y_\alpha\}_{\alpha\in\{1,2\}}$:

$$y_1:\ \ U_1 := S^n\setminus\{(0,\ldots,0,1)\}\longrightarrow\mathbb{R}^n,$$
$$(x^1,\ldots,x^n,x^{n+1})\mapsto(y_1^1,\ldots,y_1^n) := \left(\frac{x^1}{1-x^{n+1}},\ldots,\frac{x^n}{1-x^{n+1}}\right).$$

and

$$y_2:\ \ U_2 := S^n\setminus\{(0,\ldots,0,-1)\}\longrightarrow\mathbb{R}^n,$$
$$(x^1,\ldots,x^n,x^{n+1})\mapsto(y_2^1,\ldots,y_2^n) := \left(\frac{x^1}{1+x^{n+1}},\ldots,\frac{x^n}{1+x^{n+1}}\right).$$

Recall that the induced metric g of S^n from the standard Euclidean metric of $\mathbb{R}^{n+1}$ is given in local coordinates by

$$g_{ij}^{y_\alpha} = \frac{4}{(1+\sum_{i=1}^n (y_\alpha^i)^2)^2}\delta_{ij}.$$

(i) Compute the sectional curvature, Ricci curvature, and scalar curvature of the sphere S^n. (Recall we have computed the Christoffel symbols in Excercise 2.)

3. (Hyperbolic spaces)

Recall that the **hyperboloid**

$$H^n := \left\{(x^1,\ldots,x^n,x^{n+1})\in\mathbb{R}^{n+1} : \sum_{i=1}^{n}(x^i)^2 - (x^{n+1})^2 = -1, x^{n+1}>0\right\}$$

is a differentiable manifold with the following chart:

$$y: \ H^n \longrightarrow B_1(0) := \left\{ (y^1, \dots, y^n) \in \mathbb{R}^n : \sum_{i=1}^n (y^i)^2 < 1 \right\} \subset \mathbb{R}^n,$$

$$(x^1, \dots, x^n, x^{n+1}) \mapsto (y^1, \dots, y^n) := \left(\frac{x^1}{1+x^{n+1}}, \dots, \frac{x^n}{1+x^{n+1}} \right).$$

Let g be the Riemannian metric of H^n given by

$$g_{ij} = \frac{4}{(1 - \sum_{i=1}^n (y^i)^2)^2} \delta_{ij}.$$

(i) Compute the sectional curvature, Ricci curvature, and scalar curvature of the hyperbolic space H^n. (Recall we have computed the Christoffel symbols in Excercise 2.)

4. (The Second Variation Formula for length)
Let $\gamma : [a,b] \to M$ be a smooth curve and

$$F : [a,b] \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$$

be a 2-parameter variation of γ. Denote by

$$V(t) := \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

the two corresponding variational fields. Let $L(v,w) := L(\gamma_{v,w})$ be the length of the curve $\gamma_{v,w}(t) := F(t,v,w), t \in [a,b]$.

(1) Show that

$$\begin{aligned} \frac{\partial^2}{\partial w \partial v} L(v,w) = \int_a^b \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|} \Big\{ & \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle - \left\langle R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right) \frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle \\ & + \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \\ & - \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|^2} \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle \Big\} dt, \end{aligned}$$

where $\left\| \frac{\partial F}{\partial t} \right\| := \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle^{\frac{1}{2}}$.

(2) Let γ be a normal geodesic. Show that

$$\begin{aligned} \frac{\partial^2}{\partial w \partial v}_{|v=w=0} L(v,w) = & \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V, T \rangle T \langle W, T \rangle \right) dt \\ & + \langle \nabla_W V, T \rangle|_a^b, \end{aligned}$$

where $T(t) := \dot{\gamma}(t)$ is the velocity field along γ.

(3) Consider the orthogonal component $V^\perp, W^\perp$ of V, W with respect to T, that is

$$\begin{aligned} V^\perp &:= V - \langle V, T \rangle T, \\ W^\perp &:= W - \langle W, T \rangle T. \end{aligned}$$

Show that

$$\begin{aligned} \frac{\partial^2}{\partial w \partial v}_{|v=w=0} L(v,w) = & \int_a^b \left(\langle \nabla_T V^\perp, \nabla_T W^\perp \rangle - \langle R(W^\perp, T)T, V^\perp \rangle \right) dt \\ & + \langle \nabla_W V, T \rangle|_a^b, \end{aligned}$$

5. (Existence of Riemmannian metrics with positive curvature)
Consider the smooth manifold $RP^n \times RP^n$.

(i) Does there exist a Riemannian metric on $RP^n \times RP^n$ with positive sectional curvature?
(ii) Does there exist a Riemannian metric on $RP^n \times RP^n$ with positive Ricci curvature?

Remark: It is completely unknown whether $S^2 \times S^2$ has a Riemannian metric with positive sectional curvature or not. This is known as the Hopf problem.