(II $\frac{1}{2}$) Function spaces and Laplace-Beltrami operator on Riemannian Manifolds

In last Chapter, we study curves on a Rie. manifold, that is, maps $\gamma: I \subset \mathbb{R} \to M$. Particularly, we study those curves that are critical points of the Energy functional, that is, geodesics.

In this Chapter, we plan to look at maps from manifolds (i.e. functions) instead of maps into manifolds (e.g. curves). That is, we consider

$$f: M \longrightarrow \mathbb{R}.$$

Particularly, we study those functions that are critical points of Energy functional.

Recall on a Riemannian manifold (M,g), there exists a regular Borel measure vol_g ($\mathrm{vol}_g(K) < \infty$ for any compact set $K \subset M$)

Then we can consider the L^p-spaces of functions on M.

<u>Definition</u> 1. If $0 < p < \infty$, and if $f: M \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable function on M, define

$$\|f\|_p := \left(\int_M |f|^p \, d\mathrm{vol}_g \right)^{1/p}.$$

The L^p-space is .

$$L^p(\mathrm{vol}_g) = \left\{ f : M \to \mathbb{R} \cup \{\pm\infty\} \;\middle|\; \begin{array}{l} f \text{ is measurable and} \\ \|f\|_p < \infty \end{array} \right\}.$$

$f_1 \sim f_2$ if $f_1 = f_2$ a.e.

Following from the general theory, $L^p(\mathrm{vol}_g)$ is a complete metric space for $1 \le p < \infty$. (Thm 3.11 in [Rudin])

Moreover, $C_0^0(M)$ is dense in $L^p(\mathrm{vol}_g)$ for $1 \le p < \infty$. (Thm 3.14 in [Rudin])

That is, $L^p(\mathrm{vol}_g)$ $(1 \le p < \infty)$ can be considered as the completion of $C_0^0(M)$ w.r.t. the L^p-norm.

In particular, when $p=2$, we can define an inner product

$$\langle f_1, f_2 \rangle := \int_M f_1 \cdot f_2 \, d\mathrm{vol}_g, \quad \forall f_1, f_2 \in L^2(\mathrm{vol}_g).$$

That is, $L^2(\mathrm{vol}_g)$ is a Hilbert space.

We will show that on compact Rie. mflds (M, g), $L^2(\mathrm{vol}_g)$ has a complete orthonormal basis given by the eigenfunctions of the so-called Laplace-Beltrami operator.

§1. Gradient, divergence, and Laplace-Beltrami operator.

Recall that in Euclidean space $\mathbb{R}^n$, for a (smooth) function

$$f : \mathbb{R}^n \to \mathbb{R}$$

we have the gradient of f is defined as the vector field

$$\nabla f := \operatorname{grad} f := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i} \qquad (1)$$

For a (smooth) vector field $V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial x^i}$, the divergence is defined as

$$\operatorname{div}(V) = \sum_{i=1}^{n} \frac{\partial V^i}{\partial x^i} \qquad (2)$$

The Laplace operator Δ operates on (smooth) functions f via

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2} = \operatorname{div}(\operatorname{grad} f). \qquad (3)$$

- Gradient.

Now, let us consider $f \in C^\infty(M)$ for a Rie. mfld (M, g).

The gradient of f is defined as a smooth vector field $\nabla f := \operatorname{grad} f$ such that

$$\langle \operatorname{grad} f, X \rangle = X(f), \text{ for any smooth vector field } X.$$

Expression in local coordinate: Let (U, x) be a chart.

For any $X = X^i \frac{\partial}{\partial x^i}$, we have by definition

$$\langle \operatorname{grad} f, X^i \frac{\partial}{\partial x^i} \rangle = \langle (\operatorname{grad} f)^j \frac{\partial}{\partial x^j}, X^i \frac{\partial}{\partial x^i} \rangle = X^i (\operatorname{grad} f)^j g_{ji}$$

$$= X(f) = X^i \frac{\partial f}{\partial x^i} \qquad \forall\ (X^1, \dots, X^n)$$

$$\Rightarrow (\operatorname{grad} f)^j g_{ji} g^{ik} = \frac{\partial f}{\partial x^i} g^{ik} \Rightarrow (\operatorname{grad} f)^k = g^{ki} \frac{\partial f}{\partial x^i}.$$

That is, $\quad \operatorname{grad} f = g^{ij} \dfrac{\partial f}{\partial x^i} \dfrac{\partial}{\partial x^j}$.

Remark: (1) For the case $M = \mathbb{R}^n$ with the Euclidean metric $(g_{ij}) = (\delta_{ij})$, we ~~have~~ are back to (1).

(2). "The gradient vector field is vertical to the level set of a function.".

Proposition 1. Let $f \in C^\infty(M)$, c be a regular value of f. Then the vector field $\operatorname{grad} f$ is vertical to the level set $f^{-1}(c)$.

Proof: ~~Let $f \in C^\infty(M)$~~, Since c is a regular value of f, $f^{-1}(c)$ is a submanifold of M. Let

$$X \in T f^{-1}(c) \subset TM.$$

We know $X(f) = 0$ on $f^{-1}(c)$, which implies that

$$\langle \operatorname{grad} f, X \rangle = X(f) = 0 \quad \text{on } f^{-1}(c). \qquad \square$$

· Divergence.

Let us consider a smooth tangent vector field X on a Rie. mfld (M, g). Recall that the divergence of a vector field should be the "infinitesimal" changing rate of the volume element along the vector field. Let (U, x) be a chart. Then the volume element is given by the following volume n-form:

$$\Omega_0 := \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

We can measure the "infinitesimal" changing rate of Ω_0 by its Lie derivative along the vector field X: $L_X \Omega_0$.

Recall that "Cartan's magical formula": for any p-form ω on M, we have

$$L_X \omega = i(X)\, d\omega + d\, i(X)\, \omega,$$

where $i(X)$ is the interior product w.r.t. X. (i.e, the contraction of a differential form with the vector field X, that is, for any vector fields $Y_1, \dots, Y_{p-1}$, we have

$$i(X)\omega\,(Y_1, \dots, Y_{p-1}) = \omega\,(X, Y_1, \dots, Y_{p-1})\ ).$$

Therefore, we have

$$L_X \Omega_0 = i(X) \underbrace{d\Omega_0}_{\substack{\| \\ 0 \text{ since } \Omega_0 \text{ is an } n\text{-form}}} + d\,(i(X)\Omega_0) = \underbrace{d\,(\underbrace{i(X)\,\Omega_0}_{(n-1)\text{ form}})}_{n\text{-form}}.$$

Since the space of n-forms on M^n is one-dimensional, $d(i(X)\Omega_0)$ and Ω_0 differs by a function, which defined to be the divergence of X:

$$d\,(i(X)\Omega_0) := \operatorname{div}(X) \cdot \Omega_0 .$$

Remarks. (i) When we change coordinates, Ω_0 may change its sign, but this does not matter for the definition of $\mathrm{div}(X)$. (If the sign changes, it changes on both side of the equality). The global definition of $\mathrm{div}(X)$ does not require the orientability of M.

(ii). Expression in local coordinate: (U,x) be a chart.

$X = X^i \frac{\partial}{\partial x^i}$. Then we have

$$i(X)\Omega_0 = i\left(X^i \frac{\partial}{\partial x^i}\right) \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

$$= \sqrt{\det(g_{ij})}\, X^i \; i\left(\frac{\partial}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n$$

Lemma: we have $i\left(\frac{\partial}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n = (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$

Proof: Let $Y_1, \cdots, Y_{n-1}$ be any smooth vector fields. We compute

$$i\left(\frac{\partial}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n \,(Y_1, \cdots, Y_{n-1})$$

$$= dx^1 \wedge \cdots \wedge dx^n \left(\frac{\partial}{\partial x^i}, Y_1, \cdots, Y_{n-1}\right)$$

$$= \sum_{\sigma \in S(n)} (\mathrm{sgn}\,\sigma)\, dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(n)} \left(\frac{\partial}{\partial x^i}, Y_1, \cdots, Y_{n-1}\right)$$

$$= \sum_{\substack{\sigma \in S(n) \\ \sigma(1) = i}} \mathrm{sgn}(\sigma)\, dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)} (Y_1, \cdots, Y_{n-1}).$$

Notice that $\{\sigma(2), \cdots, \sigma(n)\} = \{1, 2, \cdots, \hat{i}, \cdots, n\}$.

So $\sigma(2), \ldots, \sigma(n)$ is produced by a permutation τ of $\{1, 2, \ldots, \hat{i}, \ldots, n\}$

Moreover, $\operatorname{sgn}(\sigma) = (-1)^{i-1} \operatorname{sgn} \tau$.

Therefore, $\sum_{\substack{\sigma \in S(n) \\ \sigma(1) = i}} \operatorname{sgn}(\sigma)\, dx^{\sigma(2)} \otimes \cdots \otimes dx^{\sigma(n)} (Y_1, \ldots, Y_{n-1})$

$$= \sum_{\tau \in S(n-1)} (-1)^{i-1} \operatorname{sgn} \tau \, dx^{\tau(1)} \otimes \cdots \otimes dx^{\tau(i-1)} \otimes dx^{\tau(i+1)} \otimes \cdots \otimes dx^{\tau(n)} (Y_1, \ldots, Y_{n-1})$$

$$= (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n (Y_1, \ldots, Y_{n-1}). \qquad \square$$

Applying the above lemma, we have

$$i(X)\Omega_0 = \sqrt{\det(g_{ij})} \sum_{i=1}^{n} X^i (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

Hence, we obtain

$$d(i(X)\Omega_0) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{k=1}^{n} \frac{\partial}{\partial x^k} \left(\sqrt{\det(g_{ij})} X^i\right) dx^k \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g_{ij})} X^i\right) dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g_{ij})} X^i\right) dx^1 \wedge \cdots \wedge dx^n$$

$$= \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g_{ij})} X^i\right) \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

$$= \operatorname{div}(X)\, \Omega_0$$

and $$\operatorname{div}(X) = \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x^i} \left(\sqrt{\det(g_{ij})} X^i\right).$$

Let us denote $G := \det(g_{ij})$, we have $\operatorname{div}(X) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} X^i)$

Notice that, for the case $M = \mathbb{R}^n$ with the Euclidean metric $(g_{ij}) = (\delta_{ij})$, we are back to (2).

<u>Thm 1</u>. (Divergence Theorem). Let X be a smooth vector fields with compact support on a Rie. mfld (M,g). Then
$$\int_M \operatorname{div}(X)\, dvol_g = 0.$$

<u>Proof</u>: Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite covering of M by charts, and satisfying $\#\{\alpha\in A \mid U_\alpha \cap \overline{\operatorname{supp}(X)} \neq \phi\} < \infty$. Let $\{\phi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then we have
$$X = \sum_\alpha \phi_\alpha X.$$
and
$$\int_M \operatorname{div}(X)\, dvol_g = \int_M \operatorname{div}\Big(\sum_\alpha \phi_\alpha X\Big)\, dvol_g$$
$$= \int_M \underbrace{\sum_\alpha}_{\text{finite sum}} \operatorname{div}(\phi_\alpha X)\, dvol_g = \sum_\alpha \int_M \operatorname{div}(\phi_\alpha X)\, dvol_g.$$
$$= \sum_\alpha \int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, dvol_g.$$

On each chart (U_α, x_α), we have
$$\int_{U_\alpha} \operatorname{div}(\phi_\alpha X)\, dvol_g \cancel{= \int_{U_\alpha} \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(\sqrt{G}X^i)\, dvol_g}$$
$$= \int_{x_\alpha(U_\alpha)} \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(\sqrt{G}\phi_\alpha X^i)\cdot\sqrt{G}\, dx^1\wedge\cdots\wedge dx^n = \int_{x_\alpha(U_\alpha)} \frac{\partial}{\partial x^i}(\sqrt{G}\phi_\alpha X^i)\, dx^1\cdots dx^n$$
$$= 0.$$

$\square$

· Laplace-Beltrami operator

For $f \in C^\infty(M)$, the Laplace-Beltrami operator Δ operates on f as via $\Delta f := \mathrm{div}(\mathrm{grad}\, f)$.

~~Remark~~ Expression in local chart: (U, x): we have

$$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\,(\mathrm{grad} f)^i\right) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij} \frac{\partial f}{\partial x^j}\right).$$

Remark (1). For the case $M = \mathbb{R}^n$ with the Euclidean metric $(g_{ij}) = (\delta_{ij})$, we're back to (3).

(2). By divergence theorem, we have $\int_M \Delta f \, d\mathrm{vol}_g = 0$ for any $f \in C_0^\infty(M)$.

· Green's formula.

Lemma: $\forall f \in C^\infty(M)$, and $\forall$ smooth vector field X, we have

$$\mathrm{div}(fX) = f\,\mathrm{div}(X) + \langle \mathrm{grad} f, X\rangle$$

Proof.
$$\mathrm{div}(fX) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G} f X^i)$$
$$= \frac{1}{\sqrt{G}} \frac{\partial f}{\partial x^i} \sqrt{G} X^i + f \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G} X^i)$$
$$= X(f) + f\,\mathrm{div}(X) = \langle \mathrm{grad} f, X\rangle + f\,\mathrm{div}(X). \quad \square$$

Theorem 2. (Green's formula) Let $f, h \in C^\infty(M)$, at least one of which has compact support. Then

$$\int_M f \Delta h \, d\mathrm{vol}_g = -\int_M \langle \mathrm{grad} f, \mathrm{grad}\, h\rangle \, d\mathrm{vol}_g = \int_M \Delta f \cdot h \, d\mathrm{vol}_g.$$

We often write $(f, \Delta h) = -(\operatorname{grad} f, \operatorname{grad} h) = (\Delta f, h)$ for short.

Proof: Applying the above lemma to $X = \operatorname{grad} h$, we have

$$\operatorname{div}(f \operatorname{grad} h) = f \operatorname{div}(\operatorname{grad} h) + \langle \operatorname{grad} f, \operatorname{grad} h \rangle$$
$$= f \Delta h + \langle \operatorname{grad} f, \operatorname{grad} h \rangle$$

Since $f \operatorname{grad} h$ has compact support, we can apply the divergence theorem to conclude $0 = (f, \Delta h) + \langle \operatorname{grad} f, \operatorname{grad} h \rangle$.

Similarly, we obtain $0 = (\Delta f, h) + \langle \operatorname{grad} f, \operatorname{grad} h \rangle$. $\square$.

- Energy functional.

Consider the energy ~~func~~ of a smooth function $f: M \to \mathbb{R}$ as

$$E(f) := \frac{1}{2} \int_M \langle \operatorname{grad} f, \operatorname{grad} f \rangle \, dvol_g = \frac{1}{2} \int_M \langle \nabla f, \nabla f \rangle \, dvol_g$$

Theorem 3. A smooth critical point f of the energy functional E in the sense that

$$\frac{d}{dt}\Big|_{t=0} E(f + t\eta) = 0$$

for all $\eta \in C_0^\infty(M)$ is harmonic, i.e., $\Delta f = 0$.

Proof. We compute

$$0 = \frac{d}{dt}\Big|_{t=0} \frac{1}{2} \int_M \langle \nabla(f + t\eta), \nabla f + t\eta \rangle \, dvol_g$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{1}{2} \int_M \langle \nabla f, \nabla f \rangle + 2t \langle \nabla f, \nabla \eta \rangle + t^2 \langle \nabla \eta, \nabla \eta \rangle \, dvol_g$$

$$= \int_M \langle \nabla f, \nabla \eta \rangle \, dvol_g \overset{\text{Green}}{=} - \int_M \Delta f \cdot \eta \, dvol_g \quad \forall \eta \in C_0^\infty(M).$$

Additional comments about density problem:

① for $L^p(\mu)$.

[Thm 3.14, Rudin, p69] Let X be a locally compact Hausdorff space, and let μ be a measure on a σ-algebra $\mathfrak{M}$ in X, ~~which~~ with the properties stated in Riesz Representation Thm (a positive Radon measure). For $1 \le p < \infty$, $C_0^0(X)$ is dense in $L^p(\mu)$.

If X is an open set U of $\mathbb{R}^n$ and μ the Lebesgue measure, then $C_0^\infty(X)$ is dense in $L^p(\mu)$, $1 \le p < \infty$. This is achieved by using (mollifier) regularization of functions and the fact that $C_0^0(X)$ is dense in $L^p(\mu)$. Likewise, If (M, g) is a C^∞ Rie. mfld, and μ the Rie. measure, then C_0^∞ is dense in $L^p(\mu)$, $1 \le p < \infty$.
[Prop. 3.41, p.79, 3.46, p.80, Aubin, Some Nonlinear Problems in Rie. Geometry, Springer.]

② $H^{k,p}$.

For $\mathbb{R}^n$, $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{k,p}(\mathbb{R}^n)$. But this is not true anymore for bounded open set $\Omega \subset \mathbb{R}^n$.

For complete Rie. mfld. $C_0^\infty(M)$ is dense in $H^{1,p}(M)$, $1 \le p < \infty$

But for density of $C_0^\infty(M)$ in $H^{k,p}(M)$, $k \ge 2$, one need more restrictions on (M, g). [2.4–2.7, Aubin]

Then we have $\Delta f = 0$. $\square$

Remark: The themes of harmonic functions and geodesics can be unified in the theory of harmonic maps.

§2. The Sobolev space $H^{1,2}(vol_g)$ and Rellich compactness theorem on compact Riemannian manifolds.

[Hebey] Emmanuel Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, 1999.

Let (M, g) be a smooth Rie. mfld. Let us consider the subspace $H^{1,2}(vol_g)$ of $L^2(vol_g)$ defined as below:

Set $\mathcal{C}^{1,2}(vol_g) := \left\{ u \in C^\infty(M) : \int_M |u|^2 \, dvol_g < +\infty, \int_M \langle \nabla u, \nabla u \rangle dvol_g < +\infty \right\}$

The Sobolev space $H^{1,2}(vol_g)$ is defined as the completion of $\mathcal{C}^{1,2}(vol_g)$ with respect to the norm $\|\cdot\|_{H^{1,2}}$. Here the norm is defined as

$$\|u\|_{H^{1,2}} := \left(\int_M u^2 \, dvol_g + \int_M \langle \nabla u, \nabla u \rangle \, dvol_g \right)^{1/2} \text{ for } u \in \mathcal{C}^{1,2}(vol_g).$$

Remark (1). By the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b} \le \sqrt{2(a+b)}$, $\forall a, b \ge 0$ we know the above norm is equivalent to $\left(\int_M u^2 \, dvol_g\right)^{1/2} + \left(\int_M \langle \nabla u, \nabla u \rangle dvol_g\right)^{1/2}$.

(2) ~~The~~ Any Cauchy sequence in $(\mathcal{C}^{1,2}(vol_g), \|\cdot\|_{H^{1,2}})$ is a Cauchy sequence in $(L^2(vol_g), \|\cdot\|_2)$.

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(3). Any Cauchy sequence in $(\mathcal{C}^{1,2}(vol_g), \|\cdot\|_{H^{1,2}})$ that converges to 0 in the space $(L^2(vol_g), \|\cdot\|_2)$, also converges to 0 in $(\mathcal{C}^{1,2}(vol_g), \|\cdot\|_{H^{1,2}})$.

Combining (2) and (3), we in fact obtain the natural map

$$H^{1,2} := \{\text{Cauchy sequence in } (\mathcal{C}^{1,2}(vol_g), \|\cdot\|_{H^{1,2}})\}/_{\sim} \longrightarrow \{u \in L^2 : \text{limits in } (L^2, \|\cdot\|_2) \text{ of Cauchy seq. in } (\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})\}$$

two Cauchy sequence are identified when their difference, which is again Cauchy, converges to 0.

is $\overset{(3)}{\underline{1\text{-}1}}$ and onto.

Therefore, we can identify this two spaces. Note further that for a Cauchy sequence ~~∇u_n~~ (u_n) in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$, we have

$$\begin{aligned}(|\nabla u_n| - |\nabla u_m|)^2 &= |\nabla u_n|^2 - 2|\nabla u_n||\nabla u_m| + |\nabla u_m|^2 \\ &\leq |\nabla u_n|^2 - 2\langle \nabla u_n, \nabla u_m\rangle + |\nabla u_m|^2 = \langle \nabla(u_n-u_m), \nabla(u_n-u_m)\rangle \\ &= |\nabla(u_n-u_m)|^2\end{aligned}$$

In particular, $(|\nabla u_n|)_n$ is a Cauchy sequence in $(L^2, \|\cdot\|_2)$.

And for any $u \in H^{1,2}$, $|\nabla u|$ is defined as the limit in $(L^2, \|\cdot\|_2)$ of $(|\nabla u_n|)_n$

and we can write $\|u\|_{H^{1,2}} = \left(\int_M u^2 \, dvol_g + \int_M |\nabla u|^2 \, dvol_g\right)^{1/2}$.

(4). The space $H^{1,2}(vol_g)$ is a Hilbert space with the following inner product.

$$(u,v)_{H^{1,2}} := \int_M u\cdot v \, dvol_g + \int_M \langle \nabla u, \nabla v\rangle \, dvol_g, \quad \forall\, u,v \in H^{1,2}$$

Note here $\langle \nabla u, \nabla v\rangle$ is defined via polarization, i.e.

$$\langle \nabla u, \nabla v\rangle := \frac{1}{2}\left(\langle \nabla(u+v), \nabla(u+v)\rangle - \langle \nabla u, \nabla u\rangle - \langle \nabla v, \nabla v\rangle\right)$$

(5) Density Prop: On a complete Rie. mfld, the set C_0^∞ of smooth functions with compact support is dense in $H^{1,2}(vol_g)$. [See Hebey, Thm 2.4, p.25] due to Aubin '76

Note this conclusion is not true any more for a bounded open set $U \subset \mathbb{R}^n$. Hence the completeness assumption can not be removed. For the density of $C_0^\infty(M)$ in $H^{k,p}$, $k \geq 2$, one need more restrictions on (M, g). [2.4-2.7, Aubin]

(3). (u_n) Cauchy in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$ $\Rightarrow$ u_n converge in $(L^2, \|\cdot\|_2)$

$|\nabla u_n|$ converge in $(L^2, \|\cdot\|_2)$

~~Not~~ Let $u_n \to 0$ in $(L^2, \|\cdot\|_2)$ $\Rightarrow$ $u_n \to 0$ a.e. $|\nabla u_n|$ conv. a.e.

Note that in local chart (U, x), we have $\lambda \delta_{ij} \le g_{ij} \le \Lambda \delta_{ij}$, for some $0<\lambda\le\Lambda$

Hence $u_n \in L^2(U)$, $|\nabla u_n| \in L^2(U)$.

By ~~dominated~~ dominated convergence thm (DCT), for any $\varphi \in C_0^\infty(U)$.

($\Rightarrow \varphi$, $\nabla\varphi$ are both bdded on U)

$$0 = \lim_{n\to\infty} \int_{x(U)} u_n \cdot \frac{\partial \varphi}{\partial x^i}\, dx$$

$$= -\lim_{n\to\infty} \int_{x(U)} \frac{\partial u_n}{\partial x^i} \varphi \, dx = -\int_{x(U)} \lim_{n\to\infty} \frac{\partial u_n}{\partial x^i} \varphi\, dx$$

$$\Rightarrow \lim_{n\to\infty} \frac{\partial u_n}{\partial x^i} = 0 \quad \text{a.e.}$$

$$\Rightarrow \lim_{n\to\infty} |\nabla u_n| = 0$$

That is (u_n) also converge to 0 in $(\mathcal{C}^{1,2}(\mathrm{vol}_g), \|\cdot\|_{H^{1,2}})$

Similarly, we can show if a Cauchy sequence (u_n) in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$ with limit $v \in H^{1,2}$ and $\| |\nabla v| \|_{L^2} = 0$. Then $v = \text{const.}$ a.e.

Proof: In chart (U, x), ($\lambda\delta_{ij} \le g_{ij} \le \Lambda \delta_{ij}$, for some $0<\lambda\le\Lambda$)

$\forall \varphi \in C_0^\infty(U)$. ($\varphi$, $|\nabla \varphi|$ are both bdd on $x(U)$),

$$\forall i, \lim_{n\to\infty} \int_{x(U)} u_n \circ x^{-1} \cdot \frac{\partial \varphi}{\partial x^i}\, dx \overset{DCT}{=} \int v \circ x^{-1} \frac{\partial \varphi}{\partial x^i}\, dx$$

$$= \lim_{n\to\infty} -\int_{x(U)} \frac{\partial}{\partial x^i}(u_n \circ x^{-1})\, \varphi\, dx \overset{DCT}{=} 0 \Rightarrow \text{weak derivative } D(v\circ x^{-1}) \text{ on } x(U) \text{ is zero}$$

Using mollifiers. I see Note.

$\Rightarrow$ $v \circ x^{-1}$ is constant a.e. on $x(U)$ $\Rightarrow$ v is a.e. const on M.

Therefore, we see that $H^{1,2}(vol_g)$ is a dense subspace of $L^2(vol_g)$.

Theorem (Rellich compactness Theorem). On a compact Rie. mfld (M^n, g) $H^{1,2}(vol_g)$ is compactly embedded in $L^2(vol_g)$. That is, any bounded sequence in $H^{1,2}(vol_g)$ has a convergent subsequence in $L^2(vol_g)$.

Lemma: Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $H_0^{1,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^{1,2}(\Omega)$. Then $H_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. ([Hebey] Lemma 2.5, p. 37) □

We take this chance to show how to translate results about Sobolev spaces on bounded open sets in $\mathbb{R}^n$ to that ~~of~~ in a compact Rie. mfld.

Proof of Rellich Thm: The way we ~~can achieve this~~ can reduce the thm to the Lemma is via partition of unity. Since M is compact, we can have a finite covering of M by charts

$$(U_\alpha, x_\alpha)_{\alpha=1,\cdots,N}$$

~~and~~ such that on each U_α, we have positive constants λ, Λ s.t.

$$\lambda \delta_{ij} \le g_{ij}^\alpha \le \Lambda \delta_{ij}. \qquad (*)$$

Let $(\varphi_\alpha)_{\alpha=1,\cdots,N}$ be the partition of unity subordinate to $(U_\alpha)_{\alpha=1,\cdots,N}$.

Consider a bounded sequence (u_m) in $H^{1,2}(vol_g)$.

Then $\qquad u_m = \sum_{\alpha=1}^{N} \phi_\alpha u_m$

and we obtain N functions: $u_m^\alpha := \phi_\alpha u_m \circ x_\alpha^{-1} : x_\alpha(U_\alpha) \to \mathbb{R} \cup \{\pm\infty\}$, $\alpha = 1, \cdots, N$

One can check that for any α, ~~$\phi_\alpha(u_m)$~~

(u_m^α) is a bounded sequence in $H_0^{1,2}(x_\alpha(U_\alpha))$.

By the Lemma, we know there is a subsequence $(u_{m_i}^\alpha)$ convergent in $L^2(x_\alpha(U_\alpha))$. ~~Note~~ Hence, we can find a subsequence of (u_{m_i}) of (u_m) s.t. $(u_{m_i}^\alpha)$ is Cauchy in $L^2(x_\alpha(U_\alpha))$ for any $\alpha = 1, \cdots, N$.

~~We~~ It remains to show (u_{m_i}) is Cauchy in $L^2(\mathrm{vol}_g)$.

Note first by $(*)$ that $\phi_\alpha u_m : M \to \mathbb{R} \cup \{\pm\infty\}$ is ~~bounded~~ Cauchy sequence in $L^2(\mathrm{vol}_g)$. Moreover, for any i and j, we have

$$\| u_{m_i} - u_{m_j} \|_2 = \Big\| \sum_{\alpha=1}^{N} (\phi_\alpha u_{m_i} - \phi_\alpha u_{m_j}) \Big\|_2$$
$$\leq \sum_{\alpha=1}^{N} \| \phi_\alpha u_{m_i} - \phi_\alpha u_{m_j} \|_2$$

That is, (u_{m_i}) is Cauchy in $L^2(\mathrm{vol}_g)$. $\square$

§3. The Spectrum of the Laplace-Beltrami operator on cpt Rie. Mflds.

Let (M, g) be a compact Rie. manifold. and Δ be the Laplace-Beltrami operator. An eigenfunction of Δ is a function $f \in C^2(M)$, $f \not\equiv 0$ s.t.

$$\Delta f(x) + \lambda f(x) = 0 \quad \forall x \in M.$$

for some real λ.

A λ for which such an f exists is called an <u>eigenvalue</u> of Δ. The set of all eigenvalues of Δ is called the <u>spectrum</u> of Δ.
Note by Green's formula, we have for an eigenfunction f

$$-\lambda(f,f) = (\Delta f, f) = -\int_M \langle \mathrm{grad} f, \mathrm{grad} f\rangle \, dvol_g$$

Therefore $\lambda \geq 0$.

Our aim is to show the following result, the proof of which is based on Rellich compactness theorem.

<u>Theorem</u>: Let (M,g) be a compact Rie. mfld. Then Δ has countably many eigenvalues which can be listed with multiplicity as below

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \nearrow +\infty.$$

Those eigenvalues have pairwise <u>orthonormal eigenfunctions</u> $v_0, v_1, \dots, v_m, \dots$ (i.e. $(v_m, v_\ell) = \delta_{m\ell}$) which ~~are~~ form a complete orthonormal basis in $L^2(vol_g)$. That is, for any $f \in L^2(vol_g)$, we have

$$f = \sum_{i=0}^{\infty} (f, v_i) v_i. \qquad \square$$

Now, we start <u>the proof</u>.

First observe that every constant function c satisfies $\Delta c \equiv 0$. That is, $\lambda_0 := 0$ is always an eigenvalue.

Our strategy is as follows: We first find those numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$ inductively and show that $\lambda_1 > 0$, $\lim_{m\to\infty} \lambda_m = \infty$.

Then we show the corresponding eigenfunctions form a complete orthonormal basis in ~~$H^{1,2}(vol_g)$~~ $L^2(vol_g)$. Finally, we show those eigenvalues we find are all eigenvalues.

Let us start with (we write $H^{1,2} = H^{1,2}(vol_g)$ for short)

$$\lambda_1 := \inf_{\substack{f\in H^{1,2}\setminus\{0\} \\ \int_M f\, dvol_g = 0}} \frac{\int_M \langle \nabla f, \nabla f\rangle\, dvol_g}{\int_M f^2\, dvol_g} =: \inf_{\substack{f\in H^{1,2}\setminus\{0\} \\ \int_M f\, dvol_g = 0}} \frac{(\nabla f, \nabla f)}{(f, f)}.$$

Before starting the induction argument, we first show $\lambda_1 > 0$. This is due to the following Poincaré ineq.

<u>Theorem</u> (Poincaré ineq). Let (M, g) be a compact Rie. mfld. There exists a positive constant C s.t. for any $f \in H^{1,2}$, satisfying $\int_M f\, dvol_g = 0$, we have

$$\int_M f^2\, dvol_g \leq C \int_M \langle \nabla f, \nabla f\rangle\, dvol_g.$$

<u>Proof</u>: We argue by contradiction. If the statements was false, there exists for each integers $k = 1, 2, \cdots$ ~~and~~ a function $f_k \in H^{1,2}$ with $\int_M f\, dvol_g = 0$ satisfying

$$\int_M f_k\, dvol_g > k \int_M \langle \nabla f_k, \nabla f_k\rangle\, dvol_g.$$

We renormalize by defining

$$u_k := \frac{f_k}{\|f_k\|_2}, \quad k = 1, 2, \cdots.$$

<u>Proof of Poincaré ineq.</u> Let us first reformulate our aim as follows. Denote by $\mathcal{H} := \{ u \in H^{1,2}_{(g)} : \|u\|_2 = 1, \int_M u \, dvol_g = 0 \}$

we aim at showing

$$\inf_{u \in \mathcal{H}} \int_M \langle \nabla u, \nabla u \rangle \, dvol_g > 0 .$$

Let us consider a minimizing sequence $(u_k) \in \mathcal{H}$ s.t.

$$\lim_{k \to \infty} \int_M \langle \nabla u_k, \nabla u_k \rangle \, dvol_g = \inf_{u \in \mathcal{H}} \int_M \langle \nabla u, \nabla u \rangle \, dvol_g .$$

In particular, (u_k) are bounded in $H^{1,2}$. We have

(i) by Rellich compactness theorem, that $\exists$ subsequence which converges to u_1 in $L^2(vol_g)$.

(ii) due to the reflexivity of $H^{1,2}$, that $\exists$ subsequence which converges weakly to u_2 in $H^{1,2}$.

In fact, we have $u_1 = u_2 =: u$. This is because, by Banach–Saks Thm, for a weakly convergent sequence (u_i), the sequence

$$\frac{1}{k} \sum_{i=1}^{k} u_i , \quad k = 1, 2, \ldots$$

converge strongly to the same limit as the weak limit u_2.

In particular $\left(\frac{1}{k} \sum_{i=1}^{k} u_i \right)_k$ converge strongly to u_2 in $L^2(vol_g)$.

One can check that $u_i \to u$ in $L^2(vol_g)$ implies $\frac{1}{k} \sum_{i=1}^{k} u_i$ convergent to u in $L^2(vol_g)$.)

~~TT local isometry~~.

Two more comments:

① Let $(u_n) \in (C^{1,2}, \|\cdot\|_{H^{1,2}})$ with limit $v \in H^{1,2}$ and $\|\,|\nabla v|\,\|_2 = 0$. Then $v = \text{const}$ a.e.

② If M is compact, $H^{1,2}(\mathrm{vol}_g)$ is independent of the metric g.

Proof: [Prop. 2.2 in Hebey]. If M is compact with 2 metrics g and $\tilde{g}$, there exists $C > 1$ s.t.

$$\frac{1}{C} g \le \tilde{g} \le C g$$

Therefore $H^{1,2}(\mathrm{vol}_g) = H^{1,2}(\mathrm{vol}_{\tilde{g}})$ $\square$

In particular, if we adopt $g = \sum_\alpha \phi_\alpha(g_\alpha)$ where $(\phi_\alpha)_{\alpha\in A}$ is a partition of unity subordinate to charts (U_α, x_α), and g_α is a Rie. metric on U_α, say, the Euclidean one, we can obtain equivalent $H^{1,2}$ norms.

Due to (i), we have $\int_M v \, dvol_g = 0$ and $\|v\|_2 = 1$.

Due to (ii), we have $\|v\|_{H^{1,2}} \leq \liminf_{k\to\infty} \|u_k\|_{H^{1,2}}$.

Using equivalent norms, one obtain.

$$\|v\|_2 + \int_M \langle \nabla v, \nabla v \rangle \, dvol_g \leq \liminf_{k\to\infty} \left(\|u_k\|_2 + \int_M \langle \nabla u_k, \nabla u_k \rangle \, dvol_g \right)$$

$$\Rightarrow \int_M \langle \nabla v, \nabla v \rangle \, dvol_g \leq \lim_{k\to\infty} \int_M \langle \nabla u_k, \nabla u_k \rangle \, dvol_g .$$

(We did not distinguish seq. and subseq.)

Therefore, $v \in H^{1,2}$ and $v \in \mathcal{H}$.

That is, the infimum can be achieved.

$$\int_M \langle \nabla v, \nabla v \rangle \, dvol_g = \inf_{u \in \mathcal{H}} \int_M \langle \nabla u, \nabla u \rangle \, dvol_g .$$

Notice that v can not be a constant, (since $\int_M v \, dvol_g = 0$), ($\int_M v^2 \, dvol_g = 1$)

we have $\int_M \langle \nabla v, \nabla v \rangle \, dvol_g > 0$.

In principle, one have to use the fact that $\int_M \langle \nabla v, \nabla v \rangle \, dvol_g = 0$ $\Rightarrow v = \text{const a.e.}$

The above method of argument will be used several times.

Note that Poincaré ineq. implies $\lambda_1 > 0$.

Next, we can initiate our <u>induction</u> argument.

<u>Induction</u>: ① <u>Aim</u>: λ_1 can be achieved by v_1 which is ~~an~~ eigenfunction.

Let (f_k) be a minimizing sequence in the infimum in def. of λ_1.

i.e. $\lim_{k\to\infty} \frac{(\nabla f_k, \nabla f_k)}{(f_k, f_k)} = \lambda_1$.

Notice that, we may assume that $(f_k, f_k) = \|f_k\|_2^2 = 1$ here.

Then by the same argument as in the proof of Poincaré ineq.,
we know $\exists\ v_1$ satisfying $\|v_1\| = 1$, $\int_M v_1 \, dvol_g = 0$, $v_1 \in H^{1,2}$
and $\lambda_1 = (\nabla v_1, \nabla v_1) = \dfrac{(\nabla v_1, \nabla v_1)}{(v_1, v_1)}$.

Claim 1: $-\Delta v_1 = \lambda_1 v_1$.

Proof: Denote $H_0 := \{ f \in H^{1,2} ; \int_M f \, dvol_g = 0 \}$.
Observe that, for any $\varphi \in H_0$, $t \in \mathbb{R}$

$$\frac{(\nabla(v_1 + t\varphi), \nabla(v_1 + t\varphi))}{(v_1 + t\varphi, v_1 + t\varphi)} \geq \lambda_1$$

Note that the expression in the LHS is differentiable in t,
and has a minimum at $t = 0$. So

$$0 = \frac{d}{dt}\Big|_{t=0} \frac{(\nabla(v_1 + t\varphi), \nabla(v_1 + t\varphi))}{(v_1 + t\varphi, v_1 + t\varphi)} = 2 \frac{(\nabla v_1, \nabla\varphi)(v_1, v_1) - (\nabla v_1, \nabla v_1)(v_1, \varphi)}{(v_1, v_1)^2}$$

$$= 2 \left[(\nabla v_1, \nabla\varphi) - \overbrace{(\nabla v_1, \nabla v_1)}^{\lambda_1} (v_1, \varphi) \right], \quad \forall \varphi \in H_0$$

Moreover, for the constant function 1 on M, we have $v_0 = \dfrac{1}{\sqrt{Vol(M)}}$

$$(v_1, \underset{v_0}{1}) = 0, \quad (\nabla v_1, \nabla \underset{v_0}{1}) = 0.$$

Therefore we have

$$0 = (\nabla v_1, \nabla\varphi) - \lambda_1 (v_1, \varphi) \quad \text{for any } \varphi \in H^{1,2}.$$

By regularity theory, we have $v_1 \in C^\infty(M)$, and hence

A comment about regularity theory:

Regularity is a local property. We can consider

$$0 = \int_M \langle \nabla v_1, \nabla \varphi \rangle \, dvol_g - \int_M \lambda_1 v_1 \varphi \, dvol_g$$

for $\varphi \in C_0^\infty(U)$, where (U, x) is a chart. Then the equation

reads
$$0 = \int_{x(U)} g^{ij} \frac{\partial v_1}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \sqrt{G}\, dx^1 \cdots dx^n - \int_{x(U)} \lambda_1 v_1 \varphi \sqrt{G}\, dx^1 \cdots dx^n$$

$$= \int_{x(U)} \left(g^{ij}\sqrt{G}\right) \cdot \frac{\partial v_1}{\partial x^i} \frac{\partial \varphi}{\partial x^j}\, dx^1 \cdots dx^n - \int_{x(U)} \left(\lambda_1 v \sqrt{G}\right) \varphi \, dx^1 \cdots dx^n.$$

Note that, $\exists \; 0 < \lambda \leq \Lambda$ s.t.

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n g^{ij} \sqrt{G(x)}\, \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall x \in U, \; (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$$

This means uniformly ellipticity.

Therefore, locally, v_1 is a weak solution of

$$-L v_1 = \lambda_1 v \sqrt{G}$$

where
$$L v_1 = \frac{\partial}{\partial x^j}\left(g^{ij} \sqrt{G} \frac{\partial v_1}{\partial x^i} \right)$$

Note by regularity theory, $\lambda_1 v \sqrt{G} \in H^{1,2}(U) \Rightarrow v_1 \in H^{3,2}(U')$ $\forall U' \subset\subset U$.

$\Rightarrow \lambda_1 v \sqrt{G} \in H^{3,2} \Rightarrow \cdots \Rightarrow v_1 \in H^{k,2}(U')$ $\forall k$

$\Rightarrow \forall \eta \in C_0^\infty(U)$, $\eta v_1 \in H_0^{k,2}(U)$, $\forall k$. $\xRightarrow{\text{Sobolev embedding}}$ $\eta v_1 \in C^\infty$.

$\forall x \in U$, $\exists r > 0$, with $B(x,r) \subset U$ and an $\eta \in C_0^\infty(U)$ with $\eta(x) \equiv 1$, $\forall x \in B(x,r)$

Then $\forall y \in B(x,r)$, $\eta(y) f(y) = f(y)$. [Jost, Postmodern Analysis Chap 23]

$-\Delta v_1 = \lambda v_1$ by Green's formula.

· Now assume that (λ_0°, v_0), $(\lambda_1, v_1), \dots, (\lambda_{m-1}, v_{m-1})$ has already been determined ~~iter~~ inductively, with

$$-\Delta v_i = \lambda_i v_i \quad \text{and} \quad (v_i, v_j) = \delta_{ij}, \ \forall\, i,j = \overset{0}{\cancel{1}}, \dots, m-1.$$

Denote by $H_m := \{ f \in H^{1,2} : \underbrace{(f, v_i) = 0,\ i = 1, \dots, m-1,\ (f, \cancel{1}\tfrac{1}{\sqrt{vol(\Omega)}}) = 0}_{i=0,1,2,\dots,m-1} \}$, $\frac{1}{\sqrt{vol(\Omega)}} = v_0$

and $\lambda_m := \inf\limits_{f \in H_m \setminus \{0\}} \dfrac{(\nabla f, \nabla f)}{(f, f)}$.

Observation: (i). $\lambda_m \geq \lambda_{m-1}$ due to $H_m \subset H_{m-1}$. (ii) $\left.\begin{array}{l} f_k \to f \text{ in } L^2 \\ (f_k, v_i) = 0, \forall k, \forall i = 0, \dots, m-1 \end{array}\right\} \Rightarrow (f, v_i) = 0, \forall i = 0, \dots, m-1$ (strong conv. ⇒ weak convergence)

~~(ii). H_m is a closed subspace of $H^{1,2}$ and hence a Hilbert space. This is because H_m is the orthogonal complement of a finite dimensional subspace, and is closed. (If $(f_k) \subset H_m$ converges to f, then $(f_k, v_i) = 0$. $\forall k$ $\Rightarrow (f, v_i) = 0$. So $f \in H_m$)~~

With the same argument as in the proof of Poincaré ineq., we can find $v_m \in H_m$ with $\|v_m\|_2 = 1$ s.t.

$$\lambda_m = (\nabla v_m, \nabla v_m) = \frac{(\nabla v_m, \nabla v_m)}{(v_m, v_m)}.$$

Claim 2. $-\Delta v_m = \lambda_m v_m$.

Proof. Similarly as in Claim 1. Note that in this case we'll obtain

$$0 = (\nabla v_m, \nabla \varphi) - \lambda_m (v_m, \varphi), \quad \forall \varphi \in H_m \quad (*).$$

And the fact that

$$(v_m, v_i) = 0, \quad i = 0, 1, \cdots, m-1.$$

$v_m \in H^{1,2}$ v_i satisfies this eq.

$$(\nabla v_m, \nabla v_i) = (\underbrace{\nabla v_i}_{C^\infty}, \nabla v_m) = \lambda_i (v_i, v_m) = 0$$

$$i = 0, 1, \cdots, m-1.$$

extends (*) to hold for all $\varphi \in H^{1,2}$.

Then the regularity theory implies $v_m \in C^\infty$ and, by Green's formula, $-\Delta v_m = \lambda_m v_m$. $\square$

In conclusion, we obtain countably many real numbers

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$$

<u>Lemma</u>: $\lim_{m\to\infty} \lambda_m = \infty$.

<u>Proof</u>: Otherwise, $\lim_{m\to\infty} \lambda_m = \lim_{m\to\infty} (\nabla v_m, \nabla v_m) < \infty$, we would have $(\nabla v_m, \nabla v_m) \leq K$ for all $m \in \mathbb{N}$.

Therefore, (v_m) is bounded in $H^{1,2}$. <u>Rellich compactness thm</u> implies that $\exists$ subsequence (v_{m_j}) converges in L^2, say to v.

$$\lim_{j\to\infty} \|v_{m_j} - v\|_2 = 0. \qquad (*)$$

However, for any $j \neq k$, $\|v_{m_j} - v_{m_k}\|_2^2 = (v_{m_j} - v_{m_k}, v_{m_j} - v_{m_k})$
$= (v_{m_j}, v_{m_j}) + (v_{m_k}, v_{m_k}) = 2$. This contradicts to (*). $\square$

Next, we aim at showing any $f \in L^2(M)$ can be expanded as

$$f = \sum_{i=0}^{\infty} (f, v_i) v_i \qquad (**)$$

Let us first show (**) for $f \in H^{1,2}$.

Denote by $f_m := \sum_{i=0}^{m} (f, v_i) v_i$

$\varphi_m := f - f_m$.

That is, φ_m is the orthogonal projection of f onto H_{m+1}, hence

$$(\varphi_m, v_i) = 0, \quad i = 0, 1, \dots, m$$

By definition of λ_{m+1}, we have

$$(\nabla \varphi_m, \nabla \varphi_m) \geq \lambda_{m+1} (\varphi_m, \varphi_m) \qquad (**)$$

Observe that

$$(\nabla \varphi_m, \nabla \varphi_m) = (\nabla (f - f_m), \nabla (f - f_m))$$
$$= (\nabla f, \nabla f) - 2(\nabla f, \nabla f_m) + (\nabla f_m, \nabla f_m) \qquad (*)$$

$$(\nabla f, \nabla f_m) = (\nabla \varphi_m + \nabla f_m, \nabla f_m) = (\nabla \varphi_m, \nabla f_m) + (\nabla f_m, \nabla f_m)$$

where $(\nabla \varphi_m, \nabla f_m) = \sum_{i=1}^{m} (f, v_i) \underbrace{(\nabla \varphi_m, \nabla v_i)}_{=0 \text{ since } \cancel{= (\varphi_m, \Delta v_i)} = -\lambda_i (\varphi_m, v_i) = 0.} = 0$

Hence $(\nabla f, \nabla f_m) = (\nabla f_m, \nabla f_m)$. and

(*) becomes $(\nabla \varphi_m, \nabla \varphi_m) = (\nabla f, \nabla f) - (\nabla f_m, \nabla f_m)$.

Combining with (**), we obtain

$$(\varphi_m, \varphi_m) \leq \frac{1}{\lambda_{m+1}} (\nabla \varphi_m, \nabla \varphi_m) \leq \frac{1}{\lambda_{m+1}} (\nabla f, \nabla f).$$

Recall $\lambda_{m+1} \to \infty$, we see φ_m converge to 0 in L^2. That is,

$$f = \lim_{m \to \infty} f_m = \sum_{i=1}^{\infty} (f, v_i) v_i \quad \text{in } L^2.$$

· Now, let us extend the expansion to $f \in L^2$.

We use the fact that $H^{1,2}$ is dense in L^2 (Since C_0^∞ is dense in both spaces)

Given any $f \in L^2$, $\exists\ (f_k) \subset H^{1,2}$ s.t. $f_k \longrightarrow f$ in L^2 as $k \to \infty$.

From the above argument, we know $f_k = \sum_{i=0}^{\infty} (f_k, v_i) v_i$, $\forall k$.

For f, we estimate

$$\|f - \sum_{i=0}^{N} (f, v_i) v_i\|_2 = \|f - f_k + f_k - \sum_{i=0}^{N} (f_k, v_i) v_i + \sum_{i=0}^{N} (f_k - f, v_i) v_i\|_2$$

$$\leq \|f - f_k\|_2 + \|f_k - \sum_{i=0}^{N} (f_k, v_i) v_i\|_2 + \underbrace{\|\sum_{i=0}^{N} (f_k - f, v_i) v_i\|_2}_{\leq \|f_k - f\|_2 \text{ by Bessel's ineq.}}$$

Therefore $\|f - \sum_{i=0}^{N} (f, v_i) v_i\|_2 \leq 2\|f - f_k\|_2 + \|f_k - \sum_{i=0}^{N} (f_k, v_i) v_i\|_2$.

So for any $\varepsilon > 0$, $\exists\ k = k(\varepsilon)$ large enough s.t. $\|f - f_k\|_2 < \frac{\varepsilon}{4}$.

Moreover, $\exists\ N_0 = N_0(k)$ large enough, whenever $N > N_0$, we have $\|f_k - \sum_{i=0}^{N} (f_k, v_i) v_i\|_2 < \frac{\varepsilon}{2}$

That is, for any $\varepsilon > 0$, $\exists\ N_0 = N_0(k) = N_0(k(\varepsilon))$, s.t. whenever $N > N_0$, we have

$$\|f - \sum_{i=0}^{N} (f, v_i) v_i\|_2 < \varepsilon. \qquad \square$$

Finally, let us verify that we have found all the eigenvalues. A useful fact is given as:

Claim 3. Eigenfunctions corresponding to different eigenvalues are L^2-orthonormal.

Proof. Let $u, v \not\equiv 0$ with $-\Delta u = \lambda u$, $-\Delta v = \mu v$, $\lambda \neq \mu$. Then

$$\lambda (u, v) = (\lambda u, v) = (-\Delta u, v) = (\nabla u, \nabla v) = (u, -\Delta v) = (u, \mu v) = \mu (u, v).$$

So $\lambda \neq \mu$ implies $(u, v) = 0$. $\square$

If there were an eigenvalue λ not contained in $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$ with an eigenfunction $v \not\equiv 0$. Then $v = \sum_{i=1}^{\infty} (v, v_i) v_i = 0$, contradiction. $\square$

A Remark on spectral geometry. (24)

An interesting direction is to explore the interaction between the geometry of the Rie. mfld (M,g) and the spectrum of the Laplace-Beltrami operator Δ on (M,g).

Two basic questions there are [GHL, 4.444]

i) Can we compute the spectrum of a given Rie. mfld (M,g)?

ii) Conversely, is a Rie. mfld determined up to isometry by the data of its spectrum?

It turns out the formulations of this two questions are too optimistic. In fact, the effective computation of the spectrum is impossible but in a very few cases. And, in 1964, J. Milnor constructed the first counterexample to ii), namely, a pair of isospectral, nonisometric flat tori in dimension 16. Numerous other examples of isospectral manifolds follows afterwards.

Therefore, mathematicians mainly deal with the following problems:

i)' What estimates on the spectrum can be deduced from geometric data of the manifold?

ii)' Conversely, what geometric data can be read from spectral ones?

• A). Weyl's asymptotics: Let $N(\lambda)$ be the number of eigenvalues (counted with multiplicity) that are $\le \lambda$.

Then: for $\lambda \to \infty$, $N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M) \lambda^{n/2}$

where ω_n is the volume of unit ball in $\mathbb{R}^n$, and $\sim$ means

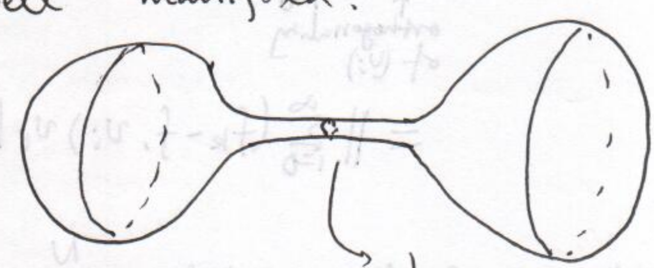
$$\lim_{\lambda\to\infty} \frac{N(\lambda) - \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M)\lambda^{n/2}}{\lambda^{n/2}} = 0$$

· B) Cheeger's ineq. In 1970, Jeff Cheeger introduce the following constant for a compact Rie. mfld.

$$h(M) = \inf \frac{\mathrm{Vol}_{n-1}(S)}{\min\{\mathrm{Vol}_n(M_1), \mathrm{Vol}_n(M_2)\}}.$$

where the infimum is taken over all $(n-1)$-dim submanifold S of M that divide M into submanifolds with boundary M_1, M_2 with $M_1 \cup M_2 = M$, $\partial M_1 = S$. $\mathrm{Vol}_{n-1}(S)$ refers to the induced of the $(d-1)$-dim. submanifold S.

"Dumbell" manifold.



The pipe is of length l and radius r

Rough idea: Let f be a function, which is equal c on the right-hand bulb, $-c$ on the left-hand bulb, and changing linear from c to $-c$ across the pipe (c is chosen s.t. $\int_M f^2 d\mathrm{vol}_g = 1$)

Then $\int_M f \, d\mathrm{vol}_g = 0$ and $|\nabla f| \approx \begin{cases} 0 & \text{bulbs,} \\ \frac{2c}{l} & \text{pipe} \end{cases}$

Hence $\lambda_1 \leq (\nabla f, \nabla f) \approx \left(\frac{2c}{l}\right)^2 \cdot 2\pi r \cdot l$. Clearly $\lambda_1 \to 0$ as $r \to 0$

<u>Theorem</u> (Cheeger). $\lambda_1 \geq \frac{h(M)^2}{4}$. $\square$