第十讲

2020年3月26日 13:51

<u>References</u>: (1) Thierry Aubin, Some nonlinear problems in Riemannian geometry, Springer.

(2) Emmanuel Hebey, Nonlinear analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, 1999.

$L^p(\mathrm{vol}_g)$ $1 \le p < \infty$

[<u>Rudin</u> Thm 3.14] X locally cpt Hausdorff space.

μ positive Radon measure.

$C_0^0(X)$ dense $L^p(\mu)$ $1 \le p < \infty$

smooth Rie. mfld.

$C_0^\infty(M)$ is dense $L^p(\mathrm{vol}_g)$ $1 \le p < \infty$

<u>Aim</u>: (M,g) compact Rie mfld.

Δ Laplace-Beltrami operator.

$\forall f, h \in C^\infty(M)$, $(\Delta f, h) = (f, \Delta h)$

An eigenfunction of Δ is a function $f \in C^2(M)$, $f \not\equiv 0$ s.t. $\boxed{\Delta f(x) + \lambda f(x) = 0}$ $\forall x \in M$. <u>for some $\lambda \in \mathbb{R}$.</u>

$\Delta f = -\lambda f$

A λ for which such an f exists is called an eigenvalue of Δ.

$$-\lambda \underbrace{(f,f)}_{\ge 0} = (\Delta f, f) \overset{\text{Green}}{=} - \underbrace{(\nabla f, \nabla f)}_{= \int_M \langle \nabla f, \nabla f \rangle \ge 0} \Rightarrow \boxed{\lambda \ge 0}$$

<u>Theorem</u>: Let (M,g) be a compact Rie. mfld. Then

Δ has countably many eigenvalues which can be listed with multiplicity as below.

$$0 = \underbrace{\lambda_0 < \lambda_1}_{} \le \lambda_2 \le \cdots \le \lambda_m \le \cdots \underbrace{\nearrow +\infty}_{}$$

Those eigenvalues have pairwise <u>orthonormal</u> eigenfcts.

$$\underset{\frac{1}{\sqrt{\mathrm{Vol}(M)}}}{\overset{}{v_0}},\ v_1, \cdots, v_m, \cdots\cdots, \quad (v_m, v_l) = \delta_{ml}$$

They form a complete orthonormal basis in $L^2(vol_g)$. That is, $\forall f \in L^2(vol_g)$, we have

$$f = \sum_{i=0}^{\infty} (f, v_i) v_i .$$

§2. The <u>Sobole space</u> $H^{1,2}(vol_g)$ and <u>Rellich compactness Theorem</u>.

Let (M, g) be a Rie. mfld. $\quad H^{1,2}(vol_g)$

$$\mathcal{C}^{1,2}(vol_g) := \Big\{ u \in C^\infty(M) : \int_M |u|^2 dvol_g < +\infty, \ \int_M \langle \nabla u, \nabla u \rangle \, dvol_g < +\infty \Big\}$$

$H^{1,2}(vol_g) :=$ the completion of $\mathcal{C}^{1,2}(vol_g)$ w.r.t. the norm $\|\cdot\|_{H^{1,2}}$ on $\mathcal{C}^{1,2}(vol_g)$.

$$\|u\|_{H^{1,2}} := \Big(\int_M u^2 dvol_g + \int_M \underbrace{\langle \nabla u, \nabla u \rangle}_{|\nabla u|^2} dvol_g \Big)^{1/2}, \ \forall u \in \mathcal{C}^{1,2}$$

<u>Remark</u>: (1) equivalent norm.

$$\|u\|_{H^{1,2}} := \|u\|_2 + \||\nabla u|\|_2 .$$

$$\left(\forall a, b \ge 0, \quad \sqrt{a+b} \le \sqrt{a} + \sqrt{b} \le \sqrt{2(a+b)} \right)$$

(2) Any Cauchy sequence in $\underline{(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})}$ is a Cauchy seq. in $(L^2, \|\cdot\|_2)$

n, m large

$\{u_n\} \quad \|u_n - u_m\|_2 + \| |\nabla(u_n - u_m)| \|_2 < \varepsilon$

Cauchy seq. in $(L, \|\cdot\|_2)$ $\quad$ n, m large

(u_n) $\quad \|u_n - u_m\|_2 + \| |\nabla(u_n - u_m)| \|_2 < \varepsilon$

$\boxed{H^{1,2}} := \{$ Cauchy sequence in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})\} / \sim$

$\sim$: two Cauchy seq. are identified when their difference sequence converges to $0 \in \mathcal{C}^{1,2}$.

i

$\{ u \in L^2 :$ u is a limit in $(L^2, \|\cdot\|_2)$ of Cauchy sequence in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})\}$

i surjective.

(3) Any Cauchy sequence in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$ that converges to 0 in $(L^2, \|\cdot\|_2)$, also converges to 0 in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$

(dominated convergence theorem) $\quad \varphi \in C_0^\infty(M)$

$u_n \longrightarrow 0$ in L^2 : $\quad \int u_n \cdot \frac{\partial \varphi}{\partial x^i} = - \int \frac{\partial u_n}{\partial x^i} \varphi$

$|\nabla u_n|$

$$\int_{U} |\nabla u_n|^2 \, dvol_g = \int_{x(U)} g^{ij} \frac{\partial u_n}{\partial x^i} \frac{\partial u_n}{\partial x^j} \sqrt{G} \, dx \quad n \to \infty$$

$$\lambda \delta_{ij} \leq g_{ij} \leq \Lambda \delta_{ij}, \quad \exists \; 0 < \lambda \leq \Lambda$$

(4) $H^{1,2} = \{ u \in L^2 :$ u is a limit of Cauchy sequence in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$ w.r.t. $\|\cdot\|_2 .\}$

$u \in H^{1,2}$: $\|u\|_{H^{1,2}} := \left(\int_M u^2 \, dvol_g + \int_M |\nabla u|^2 \, dvol_g \right)^{1/2}$

$|\nabla u| :=$ limits of $|\nabla u_n|$ in $(L^2, \|\cdot\|_2)$

$|\nabla u|$

(u_n) is a Cauchy seq. in $(\mathcal{C}^{1,2}, \|\cdot\|_{H^{1,2}})$

$\Rightarrow$ (u_n) is a Cauchy seq in $(L^2, \|\cdot\|_2)$

$(|\nabla u_n|)_n$ is a Cauchy seq. in $(L^2, \|\cdot\|_2)$

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$\|u_n - u_m\|_2 + \| |\nabla(u_n - u_m)| \|_2 < \varepsilon$

$\Rightarrow \int_M \langle \nabla(u_n - u_m), \nabla(u_n - u_m) \rangle \, dvol_g < \varepsilon$

$\Rightarrow \int_M (|\nabla u_n| - |\nabla u_m|)^2 \, dvol_g < \varepsilon$

$$(|\nabla u_n| - |\nabla u_m|)^2 = |\nabla u_n|^2 - 2|\nabla u_n||\nabla u_m| + |\nabla u_m|^2$$
$$\leq |\nabla u_n|^2 - 2\langle \nabla u_n, \nabla u_m \rangle + |\nabla u_m|^2$$
$$= \langle \nabla(u_n - u_m), \nabla(u_n - u_m) \rangle$$

(5). $H^{1,2}(vol_g)$ Hilbert space.

$\forall u, v \in H^{1,2}$

$$(u, v)_{H^{1,2}} := \int_M u \cdot v \, dvol_g + \int_M \langle \nabla u, \nabla v \rangle \, dvol_g.$$

polarization. $\langle \nabla u, \nabla v \rangle := \frac{1}{2}\{ \langle \nabla(u+v), \nabla(u+v) \rangle - \langle \nabla u, \nabla u \rangle - \langle \nabla v, \nabla v \rangle \}$

(6) Density problem:

Ω bdd open in $\mathbb{R}^n$ $\quad \overset{\circ}{H}{}^{1,2}(\Omega) \neq H^{1,2}(\Omega)$

completion of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{H^{1,2}}$

. On a complete Rie. mfld. $C_0^\infty(M)$ dense in $H^{1,2}(M)$ [Hebey, Thm 2.4. Aubin '76]

$H^{k,p}(M)$

Thm. (Rellich compact theorem). On compact Rie. mfld (M, g), $H^{1,2}(vol_g)$ is compactly embedded in

$L^2(vol_g)$, i.e., any bdd sequence in $H^{1,2}$ has a convergent subseq. in $L^2(vol_g)$.

<u>Lemma</u>: Let Ω be a bdd, open set of $\mathbb{R}^n$.
$\underline{H_0^{1,2}(\Omega)}$ = closure of $C_0^\infty(\Omega)$ in $H^{1,2}(\Omega)$.
Then $H_0^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$.

<u>Proof</u>. M cpt $\Rightarrow$ $\exists$ finite covering of M, by charts,

$$(\underline{U_\alpha, x_\alpha})_{\alpha=1,\cdots,N} \qquad (\phi_\alpha)_{\alpha=1,\cdots,N}$$

partition of unity

s.t. on each (U_α, x_α),

$$\lambda\delta_{ij} \le \underline{g_{ij}^\alpha} \le \Lambda\delta_{ij}, \text{ for some } 0<\lambda\le\Lambda.$$

Consider a bdd sequence $\underline{(u_m)}$ in $\underline{H^{1,2}(vol_g)}$

$$u_m = \sum_{\alpha=1}^N \phi_\alpha u_m$$

$u_m^\alpha =: \phi_\alpha \circ u_m \circ x_\alpha^{-1}$: $\underline{x_\alpha(U_\alpha)} \longrightarrow \mathbb{R}\cup\{\pm\infty\}$

Fix α, $(u_m^\alpha)_m$ is a bdded sequence in $H_0^{1,2}(x_\alpha(U_\alpha))$

$\Downarrow$ Lemma

$\exists$ subsequence $(u_{m_j}^\alpha)_j$ converge in $L^2(x_\alpha(U_\alpha))$

$\Rightarrow$ $\exists$ (u_{m_j}) subsequence $(u_{m_k})_k$ of (u_m), s.t.

$(u_{m_k}^\alpha)_k$ is convergent in $\underline{L^2(x_\alpha(U_\alpha))}$, $\forall\alpha=1,\cdots,N$.

$\Downarrow$

$(\phi_\alpha u_{m_k})_k$ . . . . $\underline{L^2(vol_g)}$, $\forall\alpha=1,\cdots,N$

$$\Rightarrow \|u_{m_k} - u_{m_\ell}\|_2 = \|\sum_{\alpha=1}^N(\phi_\alpha u_{m_k} - \phi_\alpha u_{m_\ell})\|_2$$

$$\Rightarrow \quad \|u_{m_k} - u_{m_\ell}\|_2 = \Big\| \sum_{\alpha=1}^{N} (\phi_\alpha u_{m_k} - \phi_\alpha u_{m_\ell}) \Big\|_2$$

$$\leq \sum_{\alpha=1}^{N} \underbrace{\|\phi_\alpha u_{m_k} - \phi_\alpha u_{m_\ell}\|_2}$$

$\Longrightarrow$ Cauchy seq. in $L^2(\mathrm{vol}_g)$. $\square$

Thm. Δ.

$$\inf_{v \in \mathcal{H}} (\nabla v, \nabla v)$$

Proof: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \nearrow +\infty$

Strategy: find those numbers $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_m \leq \cdots$

prove that they're eigenvalue.

$(\lambda_1 > 0, \quad \lambda_m \to +\infty)$

Inductively: $H^{1,2} = H^{1,2}(\mathrm{vol}_g)$

$\lambda_0 = 0.$
$$\lambda_1 := \inf_{\substack{f \in H^{1,2}\setminus\{0\} \\ (f, v_0) = 0}} \frac{\int_M \langle \nabla f, \nabla f\rangle \, d\mathrm{vol}_g}{\int_M f^2 \, d\mathrm{vol}_g}$$

$$v_0 = \frac{1}{\sqrt{\mathrm{vol}(M)}}$$

$$0 = (f, v_0) = \int_M f \frac{1}{\sqrt{\mathrm{vol}(M)}} \, d\mathrm{vol}_g$$

$$\Longleftrightarrow \int_M f \, d\mathrm{vol}_g = 0$$

$$= \inf_{f \in H_0} \frac{(\nabla f, \nabla f)}{(f, f)}$$

$$H_0 := \{ f \in H^{1,2} \setminus \{0\} : (f, v_0) = 0 \}$$

Claim. λ_1 is an eigenvalue, $\lambda_1 > 0$

$$\mathcal{H} := \{ u \in H^{1,2} \setminus \{0\}, \ \boxed{\underbrace{(u, v_0) = 0}_{\int_M u = 0}, \ \|u\|_2 = 1} \}$$

$$\lambda_1 = \inf_{f \in H_0} \frac{(\nabla f, \nabla f)}{(f, f)} = \inf_{f \in H_0} \left(\nabla \frac{f}{\|f\|_2}, \nabla \frac{f}{\|f\|_2}\right)$$

$$= \inf_{f \in \mathcal{H}} (\nabla f, \nabla f)$$

Consider a minimizing sequence, $(f_k) \subset \mathcal{H}$.

Consider a minimizing sequence, $(f_k) \subset \mathcal{H}$.

$$\underline{\lim_{k\to\infty} (\nabla f_k, \nabla f_k)} = \underline{\inf_{f\in\mathcal{H}} (\nabla f, \nabla f)}.$$

$\Rightarrow \quad (\nabla f_k, \nabla f_k) \leq M, \quad \forall k$

$\Rightarrow \quad (f_k)$ is a bdded seq. in $\underline{H^{1,2}(vol_g)}$

Rellich cpt Thm $\Rightarrow$ $\exists$ a subsequence convergent in $L^2(vol_g)$

$$f_k \longrightarrow f_1 \quad \text{in } L^2.$$

Hilbert space $\Rightarrow$ $\exists$ a subsequence weakly convergent in $H^{1,2}$

$$f_k \longrightarrow f_2 \quad \text{in } H^{1,2}$$

Banach – Saks Thm $\quad \left(\frac{1}{n}\sum_{k=1}^{n} f_k\right)_n \longrightarrow f_2$ in $H^{1,2}$

$v = f_1 = f_2$ limit

(i) L^2. $\quad \underline{(v, v_0) = 0,} \quad \underline{\|v\|_2 = 1}$

(ii) weak convergence: $\quad \|v\|_{H^{1,2}} \leq \liminf_{k\to\infty} \|f_k\|_{H^{1,2}}$

$$\cancel{\|v\|_2} + \int_M \langle \nabla v, \nabla v\rangle \, dvol_g \leq \liminf_{k\to\infty} \left(\cancel{\|f_k\|_2} + \|\,|\nabla f_k|\,\|_2\right)$$

$$\Rightarrow \quad (\nabla v, \nabla v) \leq \liminf_{k\to\infty} \underbrace{\langle \nabla f_k, \nabla f_k\rangle}$$

$$= \inf_{f\in\mathcal{H}} (\nabla f, \nabla f)$$

(i) + (ii) . $\boxed{v \in \mathcal{H}.}$

$$(\nabla v, \nabla v) \geq \lambda_1 = \inf_{f \in \mathcal{H}} (\nabla f, \nabla f) \geq^{v} (\nabla v, \nabla v)$$

$$=\qquad\qquad =$$

$$\Rightarrow \quad \lambda_1 = (\nabla v, \nabla v)$$

$\boxed{\lambda_1 > 0}$ $\quad \lambda_1 = \inf_{f \in \mathcal{H}} (\nabla f, \nabla f) = (\nabla v, \nabla v) > 0$

$$v \in \mathcal{H} : \begin{cases} \int_M v \, dvol_g = 0 \\ \int_M v^2 \, dvol_g = 1 \end{cases}$$

$\Rightarrow$ v is not a constant fct.

Thm (Poincaré ineq). (M, g) cpt Rie. mfld.

$\exists$ $C > 0$ positive const. s.t.

$\forall$ $f \in H^{1,2}(M, g)$ with $\int_M f \, dvol_g = 0$

we have

$$\int_M f^2 \, dvol_g \leq C \int_M \langle \nabla f, \nabla f \rangle \, dvol_g$$

Proof: $0 < \lambda_1 = \inf_{\substack{f \in H^{1,2} \setminus \{0\} \\ \int_M f \, dvol_g = 0}} \dfrac{(\nabla f, \nabla f)}{(f, f)}$

$$C := \frac{1}{\lambda_1}$$

$\square$

下课.