第十一讲

2020年3月31日 9:30

Comments: ① If (M,g) is compact, $H^{1,2}(vol_g)$ is independent of the choices of g.

g, $\tilde{g}$ two Rie metrics on M, M cpt mfld

Reason: $\exists\, C>1$, $\underbrace{\frac{1}{C} g \le \tilde{g} \le C g}$.

$\forall\, X,Y$ $\quad \boxed{\frac{1}{C} g(X,Y) \le \tilde{g}(X,Y) \le C g(X,Y)}$

$\forall\, f, h \in C^\infty(M)$ $\quad \langle \nabla f, \nabla h\rangle_g = g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial h}{\partial x^j}$

$\langle \nabla f, \nabla h\rangle_{\tilde{g}} = \tilde{g}^{ij} \frac{\partial f}{\partial x^i}\frac{\partial h}{\partial x^j}$

(M,g) cpt. (U_α, x_α) covering of M. chart $\{\phi_\alpha\}_\alpha$ partition of unity

$$\boxed{g = \sum_\alpha \phi_\alpha\, g_{U_\alpha}}$$

$g_{U_\alpha} \to$ Euclidean metric

$$\boxed{\lambda_1 = \inf_{\substack{f \in H^{1,2}\setminus\{0\} \\ (f, v_0) = 0}} \frac{(\nabla f, \nabla f)}{(f,f)}}$$

$v_0 = \frac{1}{\sqrt{vol(M)}}$

$\exists\, v \in H^{1,2}$ $\|v\|_2 = 1$, $(v, v_0) = 0$ s.t.

$$\lambda_1 = \underline{(\nabla v, \nabla v)} = \frac{(\nabla v, \nabla v)}{(v,v)}$$

Poincaré ineq.: $\lambda_1 > 0$

$v \in H^{1,2}$, $(\nabla v, \nabla v) = 0 \Rightarrow v \equiv$ const a.e.

$\exists\, (u_n)_n$ Cauchy in $(\mathcal{C}^{1,2}, vol_g)$, $|\nabla u_n| \to$ in L^2

$\forall\, (U,x)$ chart, $\quad \underline{\varphi \in C_0^\infty(U)} \Rightarrow \varphi, |\nabla \varphi|$ is bdd on U

$$\lim \int u_n \circ x^{-1} \cdot \frac{\partial}{\partial x^i}(\varphi\circ x^{-1})\, dx = \int v \circ x^{-1} \cdot \frac{\partial}{\partial x^i}(\varphi \circ x^{-1})\, dx$$

$$\underset{n\to\infty}{\lim} \int_{X(U)} \cdots \partial x^i \qquad \underset{X(U)}{} \ //$$

$$\lim_{n\to\infty} -\int_{X(U)} \underbrace{\frac{\partial}{\partial x^i}(u_n\circ X^{-1})}\cdot \varphi\circ X^{-1}\, dx = 0 \quad ,\forall i$$

$$\Rightarrow D(v_0\circ X^{-1}) = 0$$

weak derivative.

$$\Rightarrow v_0\circ X^{-1} \equiv \text{const a.e.} \quad \square$$

$$\begin{array}{ccccccccc} 0=\overset{0}{\overset{\shortparallel}{\lambda_0}} & < & \lambda_1 & \le & \lambda_2 & \le \cdots \le & \lambda_m & \le \cdots \nearrow +\infty \\ \wr & & \wr & & \wr & & \wr \\ v_0=\frac{1}{\sqrt{\mathrm{vol}(M)}}, & & v_1 & & v_2 & \cdots & v_m \end{array}$$

Induction: $\lambda_1 = \inf\limits_{f\in H_1\setminus\{0\}} \frac{(\nabla f,\nabla f)}{(f,f)}$, $\underline{H_1=\{f\in H^{1,2}, (f,v_0)=0\}}$

$\exists\ \underline{v\in H_1}$, $\|v\|_2=1$ s.t. $\lambda_1 = (\nabla v,\nabla v) = \frac{(\nabla v,\nabla v)}{(v,v)}$

$$\frac{(\nabla v,\nabla v)}{(v,v)} = \inf_{f\in H_1\setminus\{0\}} \frac{(\nabla f,\nabla f)}{(f,f)} \quad \|v\|_2=1$$

$$\forall \varphi\in H_1,\quad t\in\mathbb{R},\quad v+t\varphi\in H_1\setminus\{0\}$$

$$t\mapsto \underbrace{\frac{(\nabla(v+t\varphi),\nabla(v+t\varphi))}{(v+t\varphi,v+t\varphi)}} \geqslant \frac{(\nabla v,\nabla v)}{(v,v)},\ \forall t\in(-\varepsilon,\varepsilon) \quad t \text{ small}$$

$t=0$

$$\Rightarrow 0 = \frac{d}{dt}\Big|_{t=0} \frac{(\nabla(v+t\varphi),\nabla(v+t\varphi))}{(v+t\varphi,v+t\varphi)}$$

$$= \frac{2(\nabla v,\nabla\varphi)(v,v) - (\nabla v,\nabla v)2(v,\varphi)}{(v,v)^2}$$

$$= 2(\nabla v,\nabla\varphi) - 2\overbrace{(\nabla v,\nabla v)}^{\lambda_1}(v,\varphi)$$

$$\Rightarrow (\nabla v,\nabla\varphi) - \lambda_1(v,\varphi) = 0,\quad \forall\varphi\in H_1$$

$$\underbrace{\int_M \langle\nabla v,\nabla\varphi\rangle\, d\mathrm{vol}_g - \lambda_1\int_M v\varphi\, d\mathrm{vol}_g = 0,}\ \cancel{\forall\varphi\in H_1}\ \ \forall\varphi\in H^{1,2}$$

$$(\nabla v,\nabla \underbrace{v_0}_{0}) = 0,\quad (v,v_0)=0$$

Regularity theory $\Rightarrow$ $v \in C^\infty$

$$\circledast \Rightarrow \int_M (\Delta v + \lambda_1 v)\varphi = 0, \quad \forall \varphi \in \underline{C_0^\infty(M)}$$

$$\Rightarrow \Delta v + \lambda_1 v = 0 \Rightarrow -\Delta v = \lambda_1 v, \quad v \in C^\infty$$

<u>Remark</u>: (U, x) chart. $\forall \varphi \in C_0^\infty(U)$.

$$0 = \int_M \underbrace{\langle \nabla v, \nabla \varphi \rangle}\, d\mathrm{vol}_g - \int_M \lambda_1 v \underline{\varphi}\, d\mathrm{vol}_g$$

$$\Rightarrow 0 = \int_{x(U)} g^{ij} \underline{\frac{\partial v}{\partial x^i}} \frac{\partial \varphi}{\partial x^j} \sqrt{G}\, dx - \int_{x(U)} \lambda_1 v \varphi \sqrt{G}\, dx$$

$$= \int_{x(U)} \underbrace{\left(g^{ij}\sqrt{G}\right)} \frac{\partial v}{\partial x^i} \frac{\partial \varphi}{\partial x^j}\, dx - \int_{x(U)} (\lambda v \sqrt{G}) \cdot \varphi\, dx$$

$$\lambda \delta_{ij} \leq \underline{g^{ij}\sqrt{G}(x)} \leq \Lambda \delta_{ij} \quad \forall x \in U$$

$$0 < \lambda \in \Lambda$$

$$\Updownarrow$$

$\forall (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$

$$\underline{\lambda |\xi|^2 \leq g^{ij}\sqrt{G}\, \xi_i \xi_j(x) \leq \Lambda |\xi|^2}, \quad \forall x \in U$$

<u>uniformly elliptic</u>

<u>locally</u>, v is a weak solution of

$$-\underline{L} v = \underbrace{\underline{\lambda_1}\, v\sqrt{G}}$$

$$\underline{L = \frac{\partial}{\partial x^j}\left(g^{ij}\sqrt{G}\, \frac{\partial v}{\partial x^i}\right)}$$

$v \in H^{1,2}(x(U))$ $\quad \lambda_1 v\sqrt{G} \in H^{1,2}(U) \Rightarrow v_1 \in H^{3,2}(\underline{U'})$

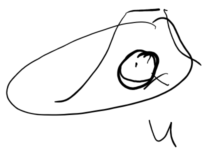
$$\underline{\forall\, U' \subset\subset U}$$

$$\Rightarrow \lambda_1 v\sqrt{G} \in H^{3,2}(\underline{U'}) \Rightarrow v \in H^{5,2} \Rightarrow \dots$$

$$\Rightarrow v \in H^{k,2}(\underline{U'}) \; \forall k \xrightarrow{\text{Sobolev embedding}} v \in C^\infty(U')$$

$\forall x \in U, \quad \exists B(x, r) \subset U.$

$\exists \eta \in C_0^\infty(U)$, s.t. $\eta|_{B(x,r)} = 1$



$\Rightarrow \eta v \in H_0^{k,2}, \forall k. \Rightarrow \eta v \in C^\infty$

$\forall y \in B(x,r), \quad \eta v = v$ $\square$

休息到 10:35.

- We assume that we have already determined

$(\lambda_0, v_0), (\lambda_1, v_1), (\lambda_2, v_2), \cdots, (\lambda_{m-1}, v_{m-1})$

$-\Delta v_i = \lambda_i v_i, \quad i = 0, 1, \cdots, m-1$

$(v_i, v_j) = \delta_{ij}, \quad \forall ij = 0, 1, \cdots, m-1$

Aim (λ_m, v_m).

Denote $H_m := \{ f \in H^{1,2} : (f, v_i) = 0, i = 0, 1, \cdots, m-1 \}$

Define $\lambda_m := \inf_{f \in H_m \setminus \{0\}} \frac{(\nabla f, \nabla f)}{(f, f)}$

$\underset{\subset H_m}{u_n} \to v_m$ in L^2

$(u_n, v_i) = 0$

$\downarrow$

$(v_m, v_i) = 0$

Rmk : . $\underset{H_{m-1}\setminus\{0\}}{\lambda_{m-1}} \le \underset{H_m \setminus\{0\}}{\lambda_m}$ ✓

Similarly . $\exists v_m \in H_m$, $\|v_m\|_2 = 1$ s.t.

$\lambda_m = (\nabla v_m, \nabla v_m) = \frac{(\nabla v_m, \nabla v_m)}{(v_m, v_m)}$.

If $v_m \in C^\infty$. $\lambda_m = (\nabla v_m, \nabla v_m) = (-\Delta v_m, v_m)$

$0 = \frac{d}{dt}\Big|_{t=0} \frac{(\nabla(v_m + t\varphi), \nabla(v_m + t\varphi))}{(v_m + t\varphi, v_m + t\varphi)}, \quad \forall \varphi \in H_m$

$\Rightarrow \quad 0 = \langle \nabla v_m, \nabla \varphi \rangle - \lambda_m (v_m, \varphi), \cancel{\forall \varphi \in H_m}$ $\forall \varphi \in H^{1,2}$

$\varphi \in \mathrm{span}\{v_0, v_1, \cdots, v_{m-1}\}$

$$\begin{cases} (v_m, v_i) = 0, & i = 0, 1, \cdots, m-1. \\ (\nabla v_m, \nabla v_i) = \boxed{(\nabla v_i, \nabla v_m) = \lambda_i (v_i, v_m)} = 0 \end{cases}$$

$v_m \in H^{1,2}$ $\quad \forall i = 0, 1, \cdots, m-1.$

$\Rightarrow v_m \in C^\infty \Rightarrow -\Delta v_m = \lambda_m v_m.$

Lemma: $\lim_{m\to\infty} \lambda_m = +\infty$

Proof: $\lambda_m = (\nabla v_m, \nabla v_m)$, $\|v_m\|_2 = 1$

If $\lim_{m\to\infty} \lambda_m < +\infty \iff \lim_{m\to\infty} \underline{\langle \nabla v_m, \nabla v_m \rangle} < \infty$

$\{v_m\}_{m=0}^{\infty}$ is a bdd sequence in $H^{1,2}$.

Rellich $\exists$ subsequence $\{v_{m_j}\}_j$ s.t.

$$v_{m_j} \longrightarrow v \text{ in } L^2$$

$\forall j,k$ $\quad \|v_{m_j} - v_{m_k}\|_2^2 = (v_{m_j} - v_{m_k}, v_{m_j} - v_{m_k})$

$= 2$. contradiction $\square$

Aim: $\forall f \in L^2(\mathrm{vol}_g)$, we have

$$\underbrace{f = \sum_{i=0}^{\infty} (f, v_i) v_i \text{ in } L^2}$$

$$\Updownarrow$$

$$\lim_{N\to\infty} \|f - \sum_{i=0}^{N} (f, v_i) v_i\|_2 = 0$$

Lemma. If $u, v \in C^\infty$ are eigenfcts s.t.

$$-\Delta u = \lambda u, \quad -\Delta v = \mu v, \quad \lambda \neq \mu.$$

Then $(u, v) = 0$

Proof. $\lambda (u,v) = (-\Delta u, v) = (\nabla u, \nabla v) = (u, -\Delta v) = \mu (u,v)$

$\lambda \neq \mu \Rightarrow (u,v) = 0$. $\square$

Suppose we have an eigenvalue $\lambda \notin \{\lambda_0, \lambda_1, \dots, \lambda_m, \dots\}$

v

$$-\Delta v = \lambda v$$

$(v, v_i) = 0$, $i = 0, 1, 2, \dots$

$$\Rightarrow v = \sum_{i=0}^{\infty} (v, v_i) v_i \equiv 0 \quad \text{contradiction} \ \square$$

Proof of Aim:

$\boxed{\forall f \in H^{1,2}}$, $f_m = \sum_{i=0}^{m} (f, v_i) v_i$, $\varphi_m = f - f_m$

$\boxed{\forall f \in H^{1,2}}$, $f_m = \sum_{i=0}^{m} (f, v_i) v_i$, $\underline{\varphi_m = f - f_m}$

orthogonal projection

Aim $\|\varphi_m\|_{L^2} \longrightarrow 0$

(φ_m, φ_m) , $\begin{cases} (\varphi_m, v_i) = 0, \ i = 0, 1, \dots, m \\ \varphi_m \in H^{1,2} \end{cases}$

$$\frac{(\nabla \varphi_m, \nabla \varphi_m)}{(\varphi_m, \varphi_m)} \geq \lambda_{m+1} \qquad \varphi_m \in H_m^{\perp}$$

$$\Rightarrow \boxed{(\varphi_m, \varphi_m) \leq \frac{1}{\lambda_{m+1}} (\nabla \varphi_m, \nabla \varphi_m)} \leq \frac{1}{\lambda_{m+1}} (\nabla f, \nabla f)$$

$\frac{1}{\lambda_{m+1}} \longrightarrow 0$ as $m \to \infty$

$\lambda_{m+1} \longrightarrow \infty$ as $m \to \infty$

? $(\nabla \varphi_m, \nabla \varphi_m) = (\nabla (f - f_m), \nabla (f - f_m))$

$= (\nabla f, \nabla f) - 2 (\nabla f, \nabla f_m) + (\nabla f_m, \nabla f_m)$

Claim: $(\nabla f, \nabla f_m) = (\nabla f_m, \nabla f_m)$

$= (\nabla (f - f_m) + \nabla f_m, \nabla f_m)$

$= \underbrace{(\nabla \varphi_m, \nabla f_m)} + (\nabla f_m, \nabla f_m)$

$(\nabla \varphi_m, \nabla f_m) = (\nabla \varphi_m, \nabla \sum_{i=0}^{m} (f, v_i) v_i)$

$= \sum_{i=0}^{m} (f, v_i) \underbrace{(\nabla \varphi_m, \nabla v_i)}_{= \lambda_i (\varphi_m, v_i) = 0} = 0$ □

$\Rightarrow (\nabla \varphi_m, \nabla \varphi_m) = (\nabla f, \nabla f) - \underbrace{(\nabla f_m, \nabla f_m)}_{\geq 0}$

$\leq (\nabla f, \nabla f)$ $\qquad f \in H^{1,2}$

$\Rightarrow \forall f \in H^{1,2}$, $f = \sum_{i=0}^{\infty} (f, v_i) v_i$.

$\forall f \in L^2$: $H^{1,2}$ is dense in L^2 (C_0^∞)

$\exists (f_k)_k \subset H^{1,2}$ s.t. $f_k \longrightarrow f$ in L^2 as $k \to \infty$

$$\|f - \sum_{i=0}^{N} (f, u_i) u_i\|_2 = \| f - f_k + f_k - \sum_{i=0}^{N} (f_k, u_i) u_i + \sum_{i=0}^{N} (f_k - f, u_i) u_i \|_2$$

$$\leq \|f - f_k\|_2 + \|f_k - \sum_{i=0}^{N} (f_k, u_i) u_i\|_2 + \underbrace{\|\sum_{i=0}^{N} (\underbrace{f_k - f}_{\in L^2}, u_i) u_i\|_2}_{\text{Bessel's inq.} \; \leq \|f_k - f\|_2}$$

$$\leq 2 \|f - f_k\|_2 + \|f_k - \sum_{i=0}^{N} (f_k, u_i) u_i\|_2 .$$

$\forall \varepsilon > 0$, $\exists\, k_0 = k_0(\varepsilon)$, s.t. $\|f - f_{k_0}\|_2 < \frac{\varepsilon}{4}$

$\exists\, N_0 = N_0(k_0(\varepsilon))$ s.t. whenever $N > N_0$

$$\| f_k - \sum_{i=0}^{N} (f_k, u_i) u_i \|_2 < \frac{\varepsilon}{2}.$$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists\, N_0 = N_0(\varepsilon)$ s.t. whenever $N > N_0$

$$\| f - \sum_{i=0}^{N} (f, u_i) u_i \|_2 < \varepsilon. \qquad \square$$

休息到11:23.

(M, g) compact

Δ Laplace-Beltrami operator.

Spectrum: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$

Spectral geometry: Spectral data $\Leftrightarrow$ geometric data

(i) Given a compact Rie. mfld (M, g), can we compute the spectrum?

(ii) Conversely, is a compact Rie. mfld determined up to isometry by the spectral data?

1964, Milnor isospectral, not isometric, flat tori in dimension 16.

Spectral geometry:

(i) estimate the spectrum via geometric data

data ... geometric data

(1) estimate the spectrum via geometric data.

(2) Conversely, read geometric data from the spectral data

ϕ (M, g) derivative of a function f along a vector field X

$$df(X) = X(f) = \langle \operatorname{grad} f, X \rangle$$

(III) Connections, Parallelism and Covariant derivative

(U, x) chart $t \mapsto x(t) = (x^1(t), \dots, x^n(t))$ in U.

$$\underline{\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\,\dot{x}^j(t)\,\dot{x}^k(t) = 0}, \quad \forall i$$

$$\dot{x}(t) = (\dot{x}^1(t), \dots, \dot{x}^n(t))$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{lk,j} + g_{jl,k} - g_{jk,l}) \qquad g_{ij}$$

$(x^i) \longrightarrow (y^\alpha)$

$$\underbrace{\Gamma^i_{jk}(x)} = \tilde{\Gamma}^\alpha_{\eta\gamma}(y(x)) \frac{\partial y^\eta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial x^i}{\partial y^\alpha} + \underbrace{\frac{\partial^2 y^\alpha}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial y^\alpha}}$$

$$\underbrace{(\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j \dot{x}^k)} = (\ddot{y}^\alpha + \tilde{\Gamma}^\alpha_{\eta\gamma}(y(x))\,\dot{y}^\eta \dot{y}^\gamma) \boxed{\frac{\partial x^i}{\partial y^\alpha}}$$

$\Rightarrow$ $(1, 0)$-tensor. $(x^i) \longrightarrow (y^\alpha)$

vector field $X = \underline{X}^i \frac{\partial}{\partial x^i} = \underline{Y}^\alpha \frac{\partial}{\partial y^\alpha}$. $\quad X^i = Y^\alpha \boxed{\frac{\partial x^i}{\partial y^\alpha}}$

$\underline{\dot{x}(t)} = (\dot{x}^1(t), \dots, \dot{x}^n(t))$ length is preserved.

geodesic $\gamma = \gamma(t)$, $\dot{\gamma}(t)$

$$\boxed{\nabla_{\dot{\gamma}(t)} \underline{\dot{\gamma}(t)}} = 0$$

§1. Affine connection.

X, Y vectorfield

directional derivative $\mathbb{R}^n$, $p \in \mathbb{R}^n$, $p \in U \subset \mathbb{R}^n$

$f \in C^\infty(U)$. $\quad v \in T_p\mathbb{R}^n$

directional derivative of f at p along v

$$D_v f = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

X C^∞ vector field $X = X^i \frac{\partial}{\partial x^i} \cong (X^1, \dots, X^n)$

$$D_v X = \underline{(D_v X^1, \dots, D_v X^n)}$$

$$= \underline{\sum_{i=1}^n (D_v X^i) \frac{\partial}{\partial x^i}}$$

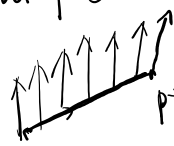
$$= \lim_{t \to 0} \frac{\overset{\in T_{p+tv}\mathbb{R}^n}{X(p+tv)} - \overset{\in T_p\mathbb{R}^n}{X(p)}}{t}$$

$$X(p+tv) - X(p) = X^i(p+tv) \left.\frac{\partial}{\partial x^i}\right|_{p+tv} - X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

$$= \left(X^i(p+tv) - X^i(p)\right) \left.\frac{\partial}{\partial x^i}\right|_p$$

Canonical identification

$X(p) =: X_0$ $\quad X_*(p+tv) =: X_t$



p $\quad p+tv$ $\quad$ Euclidean parallelism

at p $\quad v \in T_p\mathbb{R}^n$

a) $\underline{D_{\alpha v}} X = \underline{\alpha D_v X}$, $\forall \alpha \in \mathbb{R}$

b) $\underline{D_{v_1+v_2} X} = \underline{D_{v_1} X} + D_{v_2} X$, $\forall v_1, v_2$

c) $D_v(X_1 + X_2) = D_v X_1 + D_v X_2$

d) $D_v(fX) = (D_v f) X + f D_v X$.

$$\boxed{D_v \frac{\partial}{\partial x^i} \equiv 0}$$

<u>Definition</u>: (Affine connection) An affine connection ∇

on a C^∞ manifold M is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \underline{\Gamma(TM)}$$

which is denoted by $(X, Y) \xmapsto{\nabla} \boxed{\nabla_X Y}$

$\nabla(X, Y)$

satisfying $\forall X, Y, Z \in \Gamma(TM)$, $f, h \in C^\infty(M)$

(i) $\underbrace{\nabla_{fX+hY} Z} = \underline{f \nabla_X Z + h \nabla_Y Z}$ C^∞-linear

$\underline{(\nabla \text{ is tensorial in } X)}$

(ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

(∇ is $\mathbb{R}$-linear in Y)

(iii) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$.

(Leibniz rule)

<u>Rmk</u>. (0). $\underline{\nabla : \Gamma(TM) \longrightarrow \Gamma(TM) \otimes \Gamma(T^*M)}$

$Y \longmapsto \nabla Y$ (1,1)-tensor field.

$\forall X \in \Gamma(TM)$: $\underline{\nabla Y}(X) := \nabla_X Y$ covariant differentiation

df $D_v f$

(1). on $\mathbb{R}^n$, $\forall X, Y \in \Gamma(T\mathbb{R}^n)$

$$(\nabla_X Y)(p) := D_{X(p)} Y \quad , \forall p \in \mathbb{R}^n$$

$$(f \nabla_X Y)(p) \overset{?}{=} (\nabla_{fX} Y)(p) = D_{f(p) X(p)} Y = f(p) D_{X(p)} Y.$$

(2) in (U, x) chart. $X, Y \in \Gamma(T(x(U)))$.

$D_X Y$ $\qquad Y = Y^i \frac{\partial}{\partial x^i}$

$= \underline{D_X \left(Y^i \frac{\partial}{\partial x^i}\right)}$ $\qquad (x^i) \to (y^\alpha)$

$= (D_X Y^i) \frac{\partial}{\partial x^i}$ $\qquad Y = F^\alpha \frac{\partial}{\partial y^\alpha}$

in (y^α) $\quad D_X Y = (D_X F^\alpha) \frac{\partial}{\partial y^\alpha} = (D_X F^\alpha) \frac{\partial x^i}{\partial y^\alpha} \frac{\partial}{\partial x^i}$

in (y^α) $\quad D_X Y = (D_X F^\alpha)\frac{\partial}{\partial y^\alpha} = (D_X F^\alpha)\frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i}$

? $\quad D_X Y^i = (D_X F^\alpha)\frac{\partial x^i}{\partial y^\alpha}$

$$\cancel{D_X Y^i} \quad (D_X F^\alpha)\frac{\partial x^i}{\partial y^\alpha} = D_X\Big(\overbrace{Y^k \frac{\partial y^\alpha}{\partial x^k}}^{F^\alpha}\Big)\frac{\partial x^i}{\partial y^\alpha}$$

$$= D_X(Y^k)\underbrace{\frac{\partial y^\alpha}{\partial x^k}\frac{\partial x^i}{\partial y^\alpha}}_{\delta^i_k} + Y^k D_X\Big(\frac{\partial y^\alpha}{\partial x^k}\Big)\frac{\partial x^i}{\partial y^\alpha}$$

$$= D_X(Y^i) + Y^k\Big[D_X\Big(\underbrace{\frac{\partial y^\alpha}{\partial x^k}\frac{\partial x^i}{\partial y^\alpha}}_{\delta^i_k}\Big) - \frac{\partial y^\alpha}{\partial x^k} D_X\Big(\frac{\partial x^i}{\partial y^\alpha}\Big)\Big]$$

$$= D_X(Y^i) + Y^k\Big[0 - \frac{\partial y^\alpha}{\partial x^k} D_X\Big(\frac{\partial x^i}{\partial y^\alpha}\Big)\Big]$$

$$= D_X(Y^i) - \underbrace{F^\alpha D_X\Big(\frac{\partial x^i}{\partial y^\alpha}\Big)}_{?} \neq D_X(Y^i) \quad \times \quad \square$$

下课