Affine "connection": (U, x), $x = x(t)$

- $\ddot{x}^i(t) + \Gamma^i_{jk}(t)\dot{x}^j(t)\dot{x}^k(t) = 0$, $i = 1, \dots, n$
- Laplace, grad, div

Definition (Affine connection) M C^∞ mfld.

$$\nabla : \boxed{\Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)}$$

$$(X, Y) \longmapsto \nabla(X, Y) := \nabla_X Y$$

satisfying the following property. ($\forall X, Y, Z \in \Gamma(TM)$, $\forall f, g \in C^\infty(M)$)

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ (tensorial in X)

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$ ($\mathbb{R}$-linearity)

(iii) $\nabla_X (fY) = \underbrace{X(f)Y}_{\text{independent of } \nabla} + f\nabla_X Y$ (Leibniz rule)

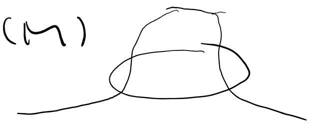
$\forall c \in \mathbb{R}$ $\nabla_X (cY) = c\nabla_X Y$

$$X(f) = \nabla_X f = X(f) = df(X)$$

Existence: M, $(U, x) \sim x(U)$

∇^U Euclidean connection on U.

Consider a $\varphi \in C_0^\infty(U) \subset C_0^\infty(M)$



$\varphi\nabla^U$ defines an affine connection on M.

$\forall X, Y \in \Gamma(TM)$

$(\varphi\nabla^U)_X Y \in \Gamma(TM)$, $(\varphi\nabla^U)_X Y(p) = \begin{cases} \varphi_\alpha \nabla^U_X Y(p) & p \in U \\ 0 & p \notin U \end{cases}$

Lemma: The set of affine connections on M form a convex set. Namely, $\nabla^{(1)}, \dots, \nabla^{(k)}$ are affine connections on M, and $f_1, \dots, f_k \in C^\infty(M)$

s.t. $\sum_i f_i \equiv 1$

Then $\sum_i f_i \nabla^{(i)}$ is also an affine connection on M.

Proof. $\boxed{\left(\sum_i f_i \nabla^{(i)}\right)_X Y (p) := \sum_i f_i(p) (\nabla^{(i)}_X Y)(p)}$

$\Rightarrow$ (i), (ii)

(iii) $h \in C^\infty(M)$, $X, Y \in \Gamma(TM)$

$$\left(\sum_i f_i \nabla^{(i)}\right)_X (hY) = \sum_i f_i \underbrace{\nabla^{(i)}_X (hY)}$$

$$= \sum_i f_i \left(X(h) Y + h \nabla^{(i)}_X Y \right)$$

$$= \underbrace{\sum_i f_i(p)}_{\parallel \atop 1} X(h) Y + h \left(\sum_i f_i \nabla^{(i)}\right)_X Y$$

$\square$

· Locality.

Proposition. For any open set $U \subset M$, if $X|_U = \tilde{X}|_U$, $Y|_U = \tilde{Y}|_U$, $X, Y, \tilde{X}, \tilde{Y} \in \Gamma(TM)$

then $\nabla_X Y|_U = \nabla_{\tilde{X}} \tilde{Y}|_U$

$\nabla_X Y(p)$ $\quad p \in U$	$\nabla_{fX} Y = f \nabla_X Y$, $\nabla_X fY = X(f) Y + f \nabla_X Y$

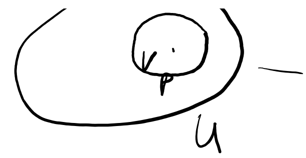
Proof. $\underbrace{\nabla_X Y|_U = \nabla_{\tilde{X}} Y|_U}_{\to\ \nabla_{X - \tilde{X}} Y|_U \equiv 0} = \overbrace{\nabla_{\tilde{X}} \tilde{Y}|_U}^{\to\ \nabla_{\tilde{X}} (Y - \tilde{Y})|_U \equiv 0}$

It's enough to show:

(i) $X|_U \equiv 0 \Rightarrow \nabla_X Y|_U \equiv 0$

(i) $X|_U \equiv 0 \Rightarrow \nabla_X Y|_U \equiv 0$

(ii) $Y|_U \equiv 0 \Rightarrow \nabla_X Y|_U \equiv 0$



Proof (i): $\forall p \in U$, we have to show $\nabla_X Y(p) = 0$

Urysohn lemma $\Rightarrow \exists\ f \in C_0^\infty(U)$ s.t. $f|_V \equiv 1$.

We check

$$X \in \Gamma(TM),\quad X|_U \equiv 0$$

$$(1-f)X \in \Gamma(TM),\quad \boxed{(1-f)X = X}$$

$$p \in U,\ X(p) \equiv 0,\quad p \notin U,\ 1-f = 1$$

$$\nabla_X Y(p) = \nabla_{(1-f)X} Y(p) = (1-f)\nabla_X Y(p)$$

$$= (1-f(p))\nabla_X Y(p) = 0 \cdot \underset{\in \Gamma(TM)}{\nabla_X Y(p)} = 0$$

$\Rightarrow \nabla_X Y|_U \equiv 0$.

(ii) $\forall p \in U$, $\exists\, p \in V \subset U$, $f \in C_0^\infty(U)$, $f|_V \equiv 1$

$$\nabla_X Y(p) = \nabla_X((1-f)Y)(p) = X(1-f)Y + (1-f)\nabla_X Y(p)$$

$$= \underbrace{X(1-f)(p)}_{\equiv 0} Y(p) + \underbrace{(1-f(p))}_{=0}\nabla_X Y(p) = 0$$

$p \in V$ $f|_V \equiv 1$, $1-f|_V \equiv 0$

$\Rightarrow Y|_U \equiv 0 \Rightarrow \nabla_X Y|_U \equiv 0$. $\square$

$X, Y \in \Gamma(TM)$, $\nabla_X Y(p)$, chart (U, x), $p \in U$

$\{\frac{\partial}{\partial x^i}, i = 1, \cdots, n\}$ coordinate vector fields

$X|_U$, $Y|_U$ $\quad X|_U = X^i \frac{\partial}{\partial x^i}$

$\quad Y|_U = \boxed{Y^j \frac{\partial}{\partial x^j}}$

$X^i \in C^\infty(U)$

$Y^j \in C^\infty(U)$

$$\nabla_X Y(p)$$

$$= X^i \nabla_{\frac{\partial}{\partial x^i}}\left(Y^j \frac{\partial}{\partial x^j}\right)(p)$$

$$= X^i\left(\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} + Y^j \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)(p)$$

$$= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j}(p) + X^i Y^j \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}(p)$$

$$= X^i \frac{\partial Y^j}{\partial x^i} \cdot \frac{\partial}{\partial x^j}(p) + X^i Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(p)$$

$$= \boxed{X(Y^j)\frac{\partial}{\partial x^j}(p) + X^i(p)\, Y^j(p) \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(p)}$$

Denote $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(p) = \begin{Bmatrix} k \\ i\,j \end{Bmatrix}(p) \frac{\partial}{\partial x^k}(p)$

Restricting to (U, x), the affine connection is uniquely determined by any n^3 C^∞ functions $\begin{Bmatrix} k \\ i\,j \end{Bmatrix}$ on U. $\square$

休息到 14:53

$\nabla_X Y(p)$ $\qquad \forall v \in T_pM$, $\nabla_v Y \in T_pM$

Proposition: If $X(p) = \tilde{X}(p)$, then $\nabla_X Y(p) = \nabla_{\tilde{X}} Y(p)$

Proof: $X \in \Gamma(TM)$, $X(p) = 0$, $\nabla_X Y(p) \equiv 0$

$$\nabla_X Y(p) = \underbrace{X(Y^k)}_{\shortparallel\, 0} \frac{\partial}{\partial x^k} + \underbrace{X^i(p)}_{\shortparallel\, 0} Y^j(p) \begin{Bmatrix} k \\ i\,j \end{Bmatrix}(p) \frac{\partial}{\partial x^k}(p)$$

$= 0$ $\square$

Proposition: Let $\gamma: (-\varepsilon, \varepsilon) \to M$ C^∞ curve.

$\gamma(0) = p$, $\dot{\gamma}(0) = v$.

If $Y(\gamma(t)) = \tilde{Y}(\gamma(t))$, $\forall t\ (-\varepsilon, \varepsilon)$.

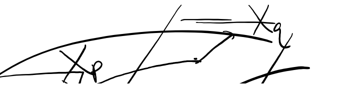
then $\nabla_v Y(p) = \nabla_v \tilde{Y}(p)$.

Proof: If $Y(\gamma(t)) \equiv 0\ \forall t \in (-\varepsilon, \varepsilon) \Rightarrow \nabla_v Y(p) = 0$

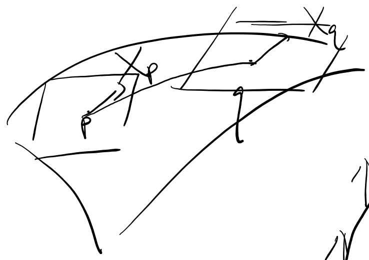
$$\nabla_v Y(p) := \underbrace{v(Y^k)}_{\shortparallel} \frac{\partial}{\partial x^k} + \underbrace{v^i Y^j(p) \begin{Bmatrix} k \\ i\,j \end{Bmatrix} \frac{\partial}{\partial x^k}(p)}_{\shortparallel\, 0}$$

$$\left.\frac{d}{dt}\right|_{t=0} Y^k(\gamma(t)) = 0 \qquad = 0. \qquad \square$$

§2. Parallelism



§2. Parallelism



Consider a C^∞ curve.

$c : [a,b] \to M$

Vector field V along c, we means

$t \in [a,b] \longmapsto V(t) \in T_{c(t)}M$

(U, x)

- If $X \in \Gamma(TM)$, ~~Define~~ Define $X(t) := X(c(t))$
- If $t \mapsto V(t)$ a v.f. along c, ... $X^i(c(t))$

C^∞ vector field along c, $c|_I \subset U$

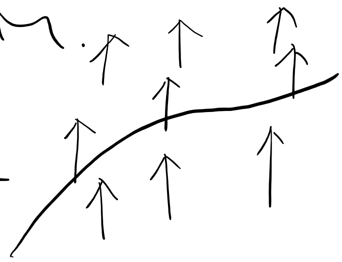
- $V(t) = \sum_{i=1}^{n} v^i(t) \frac{\partial}{\partial x^i}\Big|_{c(t)}$, $v^i \in C^\infty(I)$

$\Updownarrow$

- $\forall f \in C^\infty(M)$, $t \longmapsto V(t)(f) \in C^\infty([a,b])$.

If c is an embedding,

$V(t)$ can be extended to a vector field $\tilde{V}$ on a neighborhood of c in M.



Covariant derivative of $V(t)$ along c

$$\frac{DV}{dt} := \boxed{\nabla_{\frac{dc}{dt}} \tilde{V}(c(t))}$$

$\frac{dc}{dt} = dc\left(\frac{\partial}{\partial t}\right)$

" induced connection "

$c : [a,b] \to M$ C^∞ curve.

Proposition: (M, ∇), $\exists!$ a map

$$\mathcal{V}_c = \{C^\infty \text{ vectorfields along } c\} \to \{C^\infty \text{ vector fields along } c\}$$

$$V \longmapsto \frac{DV}{dt}$$

$V \qquad\qquad dt$

such that

(a) $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$, $\forall\ V, W \in \mathcal{V}_c$

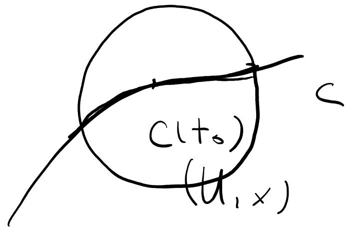
(b) $\frac{D}{dt}(fV) = \frac{df}{dt} V + f \cdot \frac{DV}{dt}$, $\quad \forall f \in C^\infty([a,b])$

(c) If $V(s) = Y(c(s))$, $\forall s \in [a,b]$ for some $\underline{Y} \in \underline{\Gamma(TM)}$,

then $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$.

Proof: Let $p_0 = c(t_0) \in M$,

Let (U, x) chart



Let us first suppose such a map exists.

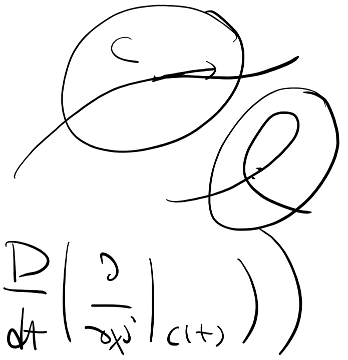
we try to show the uniqueness.

· Locality: $\frac{DV}{dt}(c(t_0))$ $\quad$ Exercise $\quad c|_I \subset U$.

$$V(t) = \sum_{j=1}^{n} v^j(t) \frac{\partial}{\partial x^j}\Big|_{c(t)} \quad t \in I$$

$$\frac{DV}{dt} = \frac{D}{dt}\left(\sum_{j=1}^{n} v^j(t) \frac{\partial}{\partial x^j}\Big|_{c(t)}\right)$$

$$\overset{(a)}{=} \sum_{j=1}^{n} \frac{D}{dt}\left(v^j(t) \frac{\partial}{\partial x^j}\Big|_{c(t)}\right)$$



$$\overset{(b)}{=} \sum_{j=1}^{n} \left(\frac{dv^j}{dt} \frac{\partial}{\partial x^j}\Big|_{c(t)} + v^j(t) \frac{D}{dt}\left(\frac{\partial}{\partial x^j}\Big|_{c(t)}\right)\right)$$

$$\overset{(c)}{=} \sum_{j=1}^{n} \left(\frac{dv^j}{dt} \frac{\partial}{\partial x^j}\Big|_{c(t)} + v^j(t) \nabla_{\frac{dc}{dt}} \frac{\partial}{\partial x^j}(c(t))\right)$$

$j=1$

in (U,x), $\quad \dfrac{dc}{dt} = \dfrac{dc^i}{dt} \cdot \dfrac{\partial}{\partial x^i}$

$c(t) = (c^1(t), \dots, c^n(t))$

$$= \sum_{j=1}^{n} \left(\frac{dv^j}{dt} \frac{\partial}{\partial x^j}\Big|_{c(t)} + v^j(t) \frac{dc^i}{dt} \boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}} (c(t)) \right)$$

$$\Rightarrow \frac{DV}{dt} = \sum_{j=1}^{n} \left(\frac{dv^j}{dt} \frac{\partial}{\partial x^j}\Big|_{c(t)} + v^j(t) \frac{dc^i}{dt} \begin{Bmatrix} k \\ ij \end{Bmatrix} \frac{\partial}{\partial x^k} (c(t)) \right)$$

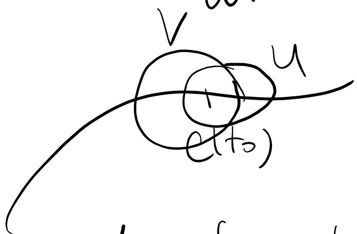
$$\Rightarrow \frac{DV}{dt} = \boxed{\sum_{k=1}^{n} \left(\frac{dv^k}{dt} + v^j(t) \frac{dc^i}{dt} \underbrace{\begin{Bmatrix} k \\ ij \end{Bmatrix}}_{\nabla} \right) \frac{\partial}{\partial x^k} (c(t))} \quad (*)$$

$(U,x) \downarrow (V,y) \quad x \to y$

If $\dfrac{DV}{dt}$ exists, then it is unique.

Existence: define $\dfrac{DV}{dt}$ in (U,x) by $(*)$

$\dfrac{DV}{dt} \in \mathcal{V}_c \qquad \dfrac{DV}{dt}(c(t_0)) \quad \dfrac{(U,x)}{(V,y)}$



Exercise: well-defined.

Verify (a), (b), (c).

□

下课.