第十三讲

2020年4月7日 9:36

$c: I \longrightarrow M$ C^∞ curve $t \longmapsto c(t)$, $t \in I$.

smooth vector field along c: $t \longmapsto V(t) \in T_{c(t)}M$, $t \in I$

Covariant derivative of $V(t)$ along c.

$$\frac{DV}{dt}$$

Chart (U, x) $\quad V = V^i(x) \frac{\partial}{\partial x^i}$

Case 1: $V \in \Gamma(TM)$ $\quad V|_c$

$$\nabla_{dc(\frac{d}{dt})} V \,(c(t)) = \nabla_{dc(\frac{d}{dt})} \left(\underline{V^i(x)} \frac{\partial}{\partial x^i} \right)(c(t))$$

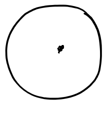
$$= \underbrace{dc\left(\frac{d}{dt}\right)(V^i)}(c(t)) \frac{\partial}{\partial x^i}(c(t)) + V^i(c(t)) \cdot \nabla_{dc(\frac{d}{dt})} \underline{\underline{\frac{\partial}{\partial x^i}}}(c(t))$$

$$= \frac{d V^i(c(t))}{dt} \frac{\partial}{\partial x^i}(c(t)) + V^i(c(t)) \nabla_{dc(\frac{d}{dt})} \frac{\partial}{\partial x^i}(c(t))$$

Case 2: $V(t)$ a vector field along c.

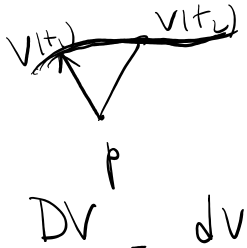
$$V(t) = \underline{V^i(t)} \frac{\partial}{\partial x^i}(c(t))$$

$$\boxed{\frac{DV}{dt} = \frac{dV^i(t)}{dt} \frac{\partial}{\partial x^i}(c(t)) + V^i(t) \nabla_{\boxed{dc(\frac{d}{dt})}} \underline{\frac{\partial}{\partial x^i}}(c(t))}$$

Remark:  $c: I \longrightarrow M$, $t \longmapsto p$ $\quad C^\infty$ curve

A vector field $V(t)$ along c:

$$V(t) = \underline{V^i(t)} \frac{\partial}{\partial x^i}(\overset{p}{\overset{\|}{c(t)}}) = V^i(t) \frac{\partial}{\partial x^i}(p)$$



$$V(t) = \overbrace{(V^1(t), \dots, V^n(t))}^{} \in \underline{T_pM}$$

$$\frac{DV}{dt} = \frac{dV^i(t)}{dt} \frac{\partial}{\partial x^i}(p) + V^i(t) \left(\nabla_{\underbrace{dc(\frac{d}{dt})}_{=0}} \frac{\partial}{\partial x^i} \right)(p)$$

$$= \frac{dV^i(t)}{} \partial$$

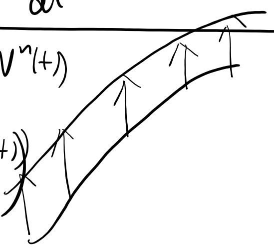
$$= \underline{\underline{\frac{dv^i(t)}{dt}\frac{\partial}{\partial x^i}(p)}} \qquad \frac{dV(t)}{dt} = \left(\frac{dv^1(t)}{dt}, \dots, \frac{dv^n(t)}{dt}\right)$$

Definition (Parallelism) (M, ∇)

A vector field V along a curve $c: I \to M$ is called parallel when $\frac{DV}{dt} = 0$, $\forall t \in I$.

Ex. $\mathbb{R}^n$, $V(t) = (v^1(t), \dots, v^n(t))$

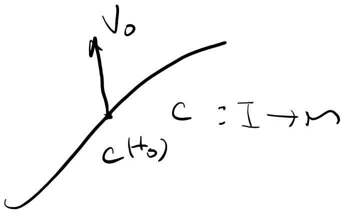
$$\frac{DV}{dt} = \left(\frac{dv^1(t)}{dt}, \dots, \frac{dv^n(t)}{dt}\right) = 0$$



Proposition: (M, ∇), $c: I \to M$

Let $V_0 \in T_{c(t_0)}M$, $t_0 \in I$.

Then $\exists!$ parallel vector field V along c s.t. $V(t_0) = V_0$.



Proof: Find a solution for $\begin{cases} \frac{DV(t)}{dt} = 0, \forall t \in I \\ V(t_0) = V_0 \end{cases}$

Let us first suppose $c|_I \subset (U, x)$, $c(t) = (c^1(t), \dots, c^n(t))$

$$\underline{V_0 = \sum_j v_0^j \frac{\partial}{\partial x^j}(c(t_0))}, \quad V(t) = \sum_j v^j(t)\frac{\partial}{\partial x^j}(c(t))$$

$$0 = \frac{DV}{dt} = \frac{dv^j}{dt}\frac{\partial}{\partial x^j}(c(t)) + v^j(t)\underbrace{\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}(c(t))}_{\shortparallel}$$

$$dc\left(\frac{d}{dt}\right) = \frac{dc^i(t)}{dt}\frac{\partial}{\partial x^i} \qquad \left(\nabla_{\frac{dc^i(t)}{dt}\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)(c(t))$$

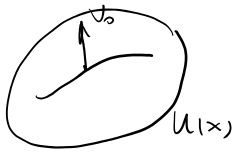
$$\Rightarrow 0 = \underline{\frac{dv^j}{dt}\frac{\partial}{\partial x^j}}(c(t)) + \underline{v^j(t)\frac{dc^i(t)}{dt}\left\{\begin{matrix} k \\ ij \end{matrix}\right\}\frac{\partial}{\partial x^k}}(c(t))$$

$$= \sum_{k=1}^n \left(\frac{dv^k}{dt} + v^j\frac{dc^i}{dt}\left\{\begin{matrix} k \\ ij \end{matrix}\right\}\right)\frac{\partial}{\partial x^k}(c(t))$$

$$\sum_{k=1}\Big(\frac{dv^k}{dt} + v^j\frac{dc^i}{dt}\begin{Bmatrix} k \\ ij\end{Bmatrix}\Big)\frac{\partial}{\partial x^k}\Big|_{c(t)}$$

$$\Longleftrightarrow \begin{cases} \dfrac{dv^k}{dt} + \displaystyle\sum_{i,j=1}^{n} v^j \underbrace{\frac{dc^i}{dt}\begin{Bmatrix} k \\ ij\end{Bmatrix}} = 0, & k=1,\dots,n \\ v^k(t_0) = v_0^k & (v^1,\dots,v^n) \end{cases}$$

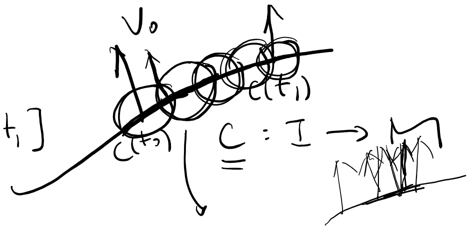
<u>ODE theory</u> $\Rightarrow$ $\exists!$ solution on I.



$c|_I \subset (U,x)$

<u>Claim</u>: $\forall t_1 \in I$, $\exists!$ parallel $V(t)$, $t\in[t_0,t_1]$

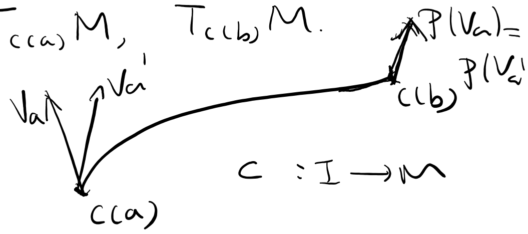
s.t. $V(t_0) = V_0$



$[t_0,t_1]$ compact, $c([t_0,t_1])$ compact in M

$\Rightarrow$ finite covering of $c([t_0,t_1])$ via charts. $\square$

Consider the tangent spaces $T_{c(a)}M$, $T_{c(b)}M$.



$$P_{c,a,b} = P: T_{c(a)}M \to T_{c(b)}M$$
$$V_a \longmapsto P(V_a)$$

<u>parallel transportation eq.</u> is linear $\Rightarrow$

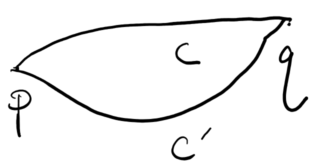
$$\begin{cases} P(V_a + W_a) = P(V_a) + P(W_a) \\ P(\lambda V_a) = \lambda P(V_a). \end{cases}$$ linear map

- P is injective. $V_a \neq V_a' \Rightarrow P(V_a) \neq P(V_a')$
  $\Uparrow$
  <u>parallel transportation equation</u> uniquess

$T_{c(a)}M$, $T_{c(b)}M$ have the same dimension.

$\Rightarrow P$ is a linear isometry between $T_{c(a)}M$, $T_{c(b)}M$.

$\underline{\nabla}$ "connection".

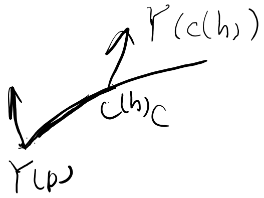


<u>$X, Y \in \Gamma(TM)$, $\nabla_X Y \in \Gamma(TM)$</u>

$Y(c(h))$

c'

$X, Y \in \Gamma(TM)$, $\nabla_X Y \in \Gamma(TM)$



Proposition: $\nabla_X Y\,(p) = \nabla_{X_p} Y\,(p)$

Let c be a C^∞ curve with $c(0) = p$, $\dot c(0) = X_p$.

Then: $\nabla_X Y(p) = \nabla_{X_p} Y = \lim_{h\to 0} \frac{1}{h}\Big(\underbrace{P^{-1}_{c,0,h}\big(Y(c(h))\big)}_{\in T_pM} - \underbrace{Y(c(0))}_{\in T_pM}\Big)$, $c(0) = p$

Proof: T_pM $\{E_i(0),\ i = 1,\dots,n\}$ a basis

define $\boxed{E_i(t) := P_{c,0,t}\big(E_i(0)\big)}$, $i = 1,\dots,n$

s.t. $\{E_i(t),\ i=1,\dots,n\}$ is a basis of $T_{c(t)}M$, $\forall t \in I$. → frame field.

$Y(c(t)) = \sum_{i=1}^n f^i(t) E_i(t)$

$\lim_{h\to 0} \frac{1}{h}\Big(\big(P_{c,0,h}\big)^{-1}\big(Y(c(h))\big) - Y(p)\Big)$

$= \lim_{h\to 0} \frac{1}{h}\Big(\big(P_{c,0,h}\big)^{-1}\Big(\sum_i f^i(h) E_i(h)\Big) - \sum_i f^i(0) E_i(0)\Big)$

$= \lim_{h\to 0} \frac{1}{h}\Big(\sum_i f^i(h) \underbrace{\big(P_{c,0,h}\big)^{-1}\big(E_i(h)\big)}_{E_i(0)} - \sum_i f^i(0) E_i(0)\Big)$

$= \sum_i \Big(\underbrace{\lim_{h\to 0} \frac{1}{h}\big(f^i(h) - f^i(0)\big)}\Big) E_i(0) = \sum_i \frac{df^i}{dt}\Big|_{t=0} E_i(0)$

$= \frac{D}{dt}\Big|_{t=0} \sum_i f^i(t) E_i(t) = \frac{DY}{dt} = \nabla_{X_p} Y$. $\square$ $\quad \nabla_{dc(\frac{d}{dt})} E_i \Rightarrow$

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M, (U, x), $f: M \to \mathbb{R}$

$\Big(\frac{\partial f}{\partial x^i}\Big) \longleftrightarrow \sum_i \frac{\partial f}{\partial x^i} dx^i = df$

$Y = y^i \frac{\partial}{\cdots}$ $\quad \Big(\frac{\partial y^i}{\cdots}\ \cdots\ y^k \Gamma^i \cdots$

$(\tilde{x}^i)$

$V = v^i \frac{\partial}{\partial x^i}$ $\boxed{\frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{kj}}$

$$\nabla_{\frac{\partial}{\partial x^j}} V = \nabla_{\frac{\partial}{\partial x^j}} \left(v^i \frac{\partial}{\partial x^i} \right) = \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial x^i} + v^i \left\{ {k \atop ij} \right\} \frac{\partial}{\partial x^k}$$

$$= \left(\frac{\partial v^k}{\partial x^j} + v^i \left\{ {k \atop ij} \right\} \right) \frac{\partial}{\partial x^k}$$

Koszul.

§3. Covariant derivative of a tensor field.

$Y \in \Gamma(TM)$, X (1,0) - tensor $\nabla_Y X$

$f : M \to \mathbb{R}$ (0,0) - tensor $\nabla_Y f := Y(f) = df(Y)$

A (r,s) - tensor field $X \in \Gamma(TM)$, $\nabla_X A$?

Proposition: (M, ∇) → affine connection over vector field

There exists a unique connection on all tensor fields : for any $r, s \in \mathbb{Z}_{\geq 0}$

$$\nabla : \Gamma(TM) \times \Gamma(\otimes^{r,s} TM) \longrightarrow \Gamma(\otimes^{r,s} TM)$$

such that. $(X, A) \longmapsto \nabla(X, A) := \nabla_X A$

~~(i) $\nabla_{fX+gY} A = f \nabla_X A + g \nabla_Y A$. (tensorial)~~

(ii) $\nabla_X (A_1 + A_2) = \nabla_X A_1 + \nabla_X A_2$ ($\mathbb{R}$-linear)

(iii) $\nabla_X (fA) = X(f) \nabla_X A + f \nabla_X A$ (Leibniz rule) $fA = f \otimes A$

and.

(iv) ∇ coincide with the given connections on $\Gamma(TM)$ and $C^\infty(M)$

(v) $\nabla_X (T_1 \otimes T_2) = \nabla_X T_1 \otimes T_2 + T_1 \otimes \nabla_X T_2$

(vi) $C(\nabla_X T) = \nabla_X (CT)$, where

(vi) $C(\nabla_X T) = \nabla_X(CT)$, where

$$C : \Gamma(\otimes^{r,s} TM) \to \Gamma(\otimes^{r-1,s-1} TM).$$

Remark. $v_1, \cdots, v_n$, $\omega^1, \cdots, \omega^n$.

$$A = \sum A^{i_1 \cdots i_r}_{j_1 \cdots j_s}\, v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_s}$$

$$CA = \sum A^{i_1 \cdots i_r}_{j_1 \cdots j_s}\, \omega^{j_1}(v_{i_r})\, v_{i_1} \otimes \cdots \otimes v_{i_{r-1}} \otimes \omega^{j_2} \otimes \cdots \otimes \omega^{j_s}.$$

Proof: $\nabla_X A$. Suppose ∇ exists.

$\nabla_X A^{i_1 \cdots i_r}_{j_1 \cdots j_s}$, $\nabla_X v_{i_1}$, $\nabla_X \omega^{j_1}$

? $\nabla_X \omega$, for $\omega \in \Omega^1(M) = \Gamma(T^*M)$

$\forall\, Y \in \Gamma(TM)$, $(\nabla_X \omega)(Y) = ?$, $\forall\, Y \in \Gamma(TM)$

$\omega(Y) \in C^\infty(M)$.

$$\nabla_X(\omega(Y)) = X(\omega(Y)) = \nabla_X(C(Y \otimes \omega))$$

$$\overset{(vi)}{=} C(\nabla_X(Y \otimes \omega)) \overset{(v)}{=} C(\nabla_X Y \otimes \omega + Y \otimes \nabla_X \omega)$$

$$= \omega(\nabla_X Y) + (\nabla_X \omega)(Y)$$

$$\Rightarrow \boxed{(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)}$$

$\Rightarrow$ uniqueness.

Existence: $\nabla_X f$, $\nabla_X V$, $\nabla_X \omega$ def

$\Rightarrow \nabla_X A \overset{\text{def}}{=}$ $\boxed{}$

Exercise. $\square$

Remark (1) $\nabla_X f$, $\nabla_X V$, $\nabla_X \omega$

Claim: $\nabla_{fX+gY}\omega = f\nabla_X\omega + g\nabla_Y\omega$

$\forall\, f,g \in C^\infty(M),\ X,Y\in\Gamma(TM)$.

Proof: $\forall\, Z\in\Gamma(TM)$,

$$(fX+gY)\big(\omega(Z)\big) = \nabla_{fX+gY}\,\omega(Z)$$

$$= \underbrace{\left(\nabla_{fX+gY}\,\omega\right)(Z)} + \omega\left(\nabla_{fX+gY} Z\right)$$

$$\|$$

$$f\,X(\omega(Z)) + g\,Y(\omega(Z))$$

$$= f\,(\nabla_X\omega)(Z) + f\,\omega(\nabla_X Z) + g\,(\nabla_Y\omega)(Z) + g\cdot\omega(\nabla_Y Z)$$

$$= \underbrace{\left(f(\nabla_X\omega) + g(\nabla_Y\omega)\right)}(Z) + \left(f\,\omega(\nabla_X Z) + g\,\omega(\nabla_Y Z)\right)$$

$$\|$$

$$\omega\big(\underbrace{f\nabla_X Z + g\nabla_Y Z}\big)$$

$$\|$$

$$\omega\left(\nabla_{fX+gY} Z\right)$$

□

Remark. (locality)

$\nabla_X A\,(p)$ → $\boxed{\nabla_X A(p) = \nabla_{X_p} A}$

U $\quad p\in U$

$\nabla_{(1-f)X} A \qquad \nabla_X (1-f)A$

In a chart (U,x) $\quad X = X^i\frac{\partial}{\partial x^i}$, $\cancel{A=}\ \omega = \omega_i dx^i$

$$\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right), \quad \left(\nabla_{\frac{\partial}{\partial x^i}} dx^j\right)\left(\frac{\partial}{\partial x^k}\right)$$

$$\|$$

$$\left\{ {k \atop ij} \right\}\frac{\partial}{\partial x^k} \qquad = \frac{\partial}{\partial x^i}\Big(\underbrace{dx^j\left(\frac{\partial}{\partial x^k}\right)}_{\delta^j_k}\Big) - dx^j\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right)$$

$$= 0 - dx^j\left(\left\{ {l \atop ik} \right\}\frac{\partial}{\partial x^l}\right)$$

$$= -\left\{ {l \atop ik} \right\} \delta^j_l = -\left\{ {j \atop ik} \right\}$$

$$\Rightarrow \boxed{\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\left\{ {j \atop ik} \right\} dx^k} \quad n^3$$

Given n^3 C^∞ fct. $\left\{ {i \atop jk} \right\} \Rightarrow \nabla_X A$

in local

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Parallel transportation

V, W two vector spaces

$\varphi : V \longrightarrow W$ isomorphism

$\Rightarrow$ ~~induced~~ induces an isomorphism

$$\varphi^* : W^* \longrightarrow V^*$$

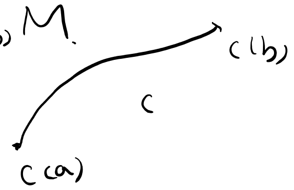
$$\alpha \longmapsto \varphi^*(\alpha), \quad \text{where } \forall v \in V \quad \varphi^*(\alpha)(v) := \alpha(\varphi(v))$$

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^r)$$

$$= \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}\alpha^r$$

$\Rightarrow \tilde{\varphi} : \otimes^{r,s} V \longrightarrow \otimes^{r,s} W$ linear isomorphism

$$\tilde{P}_{c,a,b} : \otimes^{r,s} T_{c(a)} M \longrightarrow \otimes^{r,s} T_{c(b)} M.$$



Define $\nabla_X A \in \Gamma(\otimes^{r,s} TM)$

s.t. $\nabla_X A(p) := \nabla_{X_p} A := \lim_{h \to 0} \frac{1}{h} \left(\tilde{P}_{c,0,h}^{-1}(A(c(h))) - A(p) \right)$

where c is a C^∞ curve with $c(0) = p$, $\dot{c}(0) = X_p$

Verify that $\nabla_X A$ satisfies all the properties

$T_{c(0)}M = T_pM$ $\quad$ a basis $\{E_i(0)\}_{i=1}^n$

Define $E_i(h) := \mathcal{P}_{c,0,h}(E_i(0))$ $\quad$ a frame field

Let $\{\omega^i(0)\}_{i=0}^n$ is a basis of T_p^*M

Define: $\omega^i(h) := (\mathcal{P}^*_{c,0,h})^{-1}(\omega^i(0))$, $i=1,\dots,n$

Claim: $\omega^i(h)(E_j(h)) = \delta^i_j$, $\forall h$, $\forall i,j$

Proof: $\mathcal{P}^*_{c,0,h} : T^*_{c(h)}M \longrightarrow T^*_{c(0)}M$

$$\omega^i(h) \longmapsto \mathcal{P}^*_{c,0,h}(\omega^i(h))$$

$\forall v(0) \in T_{c(0)}M$, we have by def.

$$\mathcal{P}^*_{c,0,h}(\omega^i(h))(v(0)) := \omega^i(h)(\mathcal{P}_{c,0,h}(v(0)))$$

$$\begin{aligned}\omega^i(h)(E_j(h)) &= \omega^i(h)(\mathcal{P}_{c,0,h}(E_j(0))) \\ &= \mathcal{P}^*_{c,0,h}(\omega^i(h))(E_j(0)) \\ &= \omega^i(0)(E_j(0)) = \delta^i_j. \quad \square\end{aligned}$$

$$\begin{cases} V(h) = \sum_{i=1}^n f^i(h)\, E_i(h) \\ \omega(h) = \sum_{j=1}^n g_j(h)\, \omega^j(h) \end{cases} \quad \text{parallel.}$$

$\nabla_X A$. $\quad \boxed{\nabla_X(V\otimes\omega) = \nabla_X V \otimes \omega + V \otimes \nabla_X \omega}$

$$\Rightarrow \frac{d}{dh}(f^i(h)\, g_j(h)) = \frac{df^i}{dh} g_j + f^i \frac{dg_j}{dh}$$

$$\nabla_X C(V\otimes\omega) \overset{?}{=} C(\nabla_X(V\otimes\omega))$$

$\boxed{\text{Excercise}}$

$\eta \in \Omega^1(M)$, $\forall Y \in \Gamma(TM)$

$$(\nabla_{X_p}\eta)(Y) = X_p(\eta(Y)) - \eta(\nabla_{X_p}Y) \quad \square$$

§4. Levi-Civita (Riemannian) connection

$x(t) = (x^1(t), \dots, x^n(t))$ geodesic

$$\ddot{x}^i(t) + \Gamma^i_{jk}\,\dot{x}^j(t)\,\dot{x}^k(t) = 0, \quad i = 1, \dots, n$$

$$\dot{x}(t) = (\dot{x}^1(t), \dots, \dot{x}^n(t)) = \dot{x}^i(t)\frac{\partial}{\partial x^i}$$

$$\frac{D\dot{x}}{dt} = \frac{d}{dt}(\dot{x}^i(t))\frac{\partial}{\partial x^i} + \dot{x}^i(t)\,\nabla_{\underbrace{dx(\frac{d}{dt})}_{=\frac{dx^j}{dt}\frac{\partial}{\partial x^j}}}\frac{\partial}{\partial x^i}$$

$$= \ddot{x}^i(t)\frac{\partial}{\partial x^i} + \dot{x}^i(t)\,\dot{x}^j(t)\left\{ {k \atop ji} \right\}\frac{\partial}{\partial x^k}$$

$$= \left(\ddot{x}^k(t) + \dot{x}^i(t)\,\dot{x}^j(t)\left\{ {k \atop ji} \right\}\right)\frac{\partial}{\partial x^k}$$

If $\boxed{\left\{ {k \atop ji} \right\} = \Gamma^k_{ji}}$, then for a geodesic, $\frac{D\dot{x}}{dt} = 0$

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