第十四讲

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Levi-Civita connection.

Hope: · Find an affine connection ∇ on M, such that the geodesic equation can be reformulated as

$\gamma = \gamma(t)$. $\boxed{\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0}$

(local charts: (U, x), Find an affine connection ∇ s.t. ∇ is locally determined by $\boxed{\Gamma^i_{jk}}$, $i, j, k = 1, \cdots, n$)

Koszul

Recall: $\gamma = \gamma(t)$ geodesic, $\boxed{g(\dot\gamma(t), \dot\gamma(t)) \equiv \text{const}}$, $\forall t$

$$0 \equiv \frac{d}{dt}\left(g(\dot\gamma(t), \dot\gamma(t)) \circ \gamma(t)\right) = d\gamma\left(\frac{d}{dt}\right) \cdot \left(g(\dot\gamma(t), \dot\gamma(t))\right)$$

$$\equiv \left(\nabla_{d\gamma(\frac{d}{dt})} g\right)(\dot\gamma(t), \dot\gamma(t)) + 2g\left(\boxed{\nabla_{d\gamma(\frac{d}{dt})} \dot\gamma(t)}, \dot\gamma(t)\right)$$

$= 0$

Hope. Find ∇, s.t.

If such an affine connection ∇ exists, then

$$0 = \left(\nabla_{d\gamma(\frac{d}{dt})} g\right)(\dot\gamma(t), \dot\gamma(t))$$

Definition: We say ∇ is compatible with g, if the Rie. metric tensor g is parallel.

(Recall g parallel, $\nabla_X g = 0$, $\forall X \in \Gamma(TM)$)

If ∇ is compatible with g, then

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

· Consider an affine connection ∇, $\boxed{\nabla \text{ compatible with } g}$

Question: In a chart (U, x)

$$\nabla \longleftrightarrow \left\{ {i \atop jk} \right\} \qquad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \left\{ {k \atop ij} \right\} \frac{\partial}{\partial x^k}$$

$$\left\{ {i \atop jk} \right\} \overset{?}{=} \Gamma^i_{jk}.$$

$$\frac{\partial}{\partial x^k} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij,k}$$

$$= g\left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right)$$

$$= g\left(\left\{ {l \atop ki} \right\}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \left\{ {l \atop kj} \right\}\frac{\partial}{\partial x^l}\right)$$

$$= g\left(\left\{ {l \atop ki} \right\}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \left\{ {l \atop kj} \right\}\frac{\partial}{\partial x^l}\right)$$

$$= \left\{ {l \atop ki} \right\} g_{lj} + \left\{ {l \atop kj} \right\} g_{il}$$

$$\Rightarrow \begin{cases} g_{ij,k} = g_{lj}\left\{ {l \atop ki} \right\} + g_{il}\left\{ {l \atop kj} \right\} & (*) \\ g_{jk,i} = g_{lk}\left\{ {l \atop ij} \right\} + g_{jl}\left\{ {l \atop ik} \right\} \\ g_{ki,j} = g_{li}\left\{ {l \atop jk} \right\} + g_{kl}\left\{ {l \atop ji} \right\} \end{cases}$$

$i \to j \to k$, $k \to k \to i$, $\to i \to j$

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

$$g_{ij,k} + g_{jk,i} - g_{ki,j} = g_{lj}\left(\left\{ {l \atop ki} \right\} + \left\{ {l \atop ik} \right\}\right) + g_{il}\underbrace{\left(\left\{ {l \atop kj} \right\} - \left\{ {l \atop jk} \right\}\right)}_{0}$$

$$+ g_{lk}\underbrace{\left(\left\{ {l \atop ij} \right\} - \left\{ {l \atop ji} \right\}\right)}_{0}$$

- symmetric $\left\{ {l \atop jk} \right\} = \left\{ {l \atop kj} \right\}$

$$\Rightarrow \underbrace{\frac{1}{2} g^{pj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)}\ = 2 g_{lj} \left\{ {l \atop ki} \right\} g^{jp}$$

$$= \left\{ {p \atop ki} \right\}$$

$$\Rightarrow \left\{ {p \atop ki} \right\} = \Gamma^p_{ki} \quad !$$

Restriction: ① $\nabla_X g = 0, \ \forall X \in \Gamma(TM)$

② $\left\{ {i \atop jk} \right\} = \left\{ {i \atop kj} \right\}$ symmetric

$$\Downarrow$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \left\{ {i \atop jk} \right\} \frac{\partial}{\partial x^i}$$

$$\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} = \left\{ {i \atop kj} \right\} \frac{\partial}{\partial x^i}$$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}$$

$(U, x) \Rightarrow \forall X, Y$ v.f on U

$$\nabla_X Y = \nabla_Y X$$

$$D(X,Y) =: \nabla_X Y - \nabla_Y X \equiv 0 \ ?$$

$$D(fX, Y) \overset{?}{=} f\, D(X,Y), \quad \forall f \in C^\infty(M) \quad \text{not a tensor!}$$

$$\parallel$$

$$\nabla_{fX} Y - \nabla_Y (fX) = f\nabla_X Y - Y(f)X - f\nabla_Y X$$

Definition. Torsion tensor $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$,

$\forall X, Y \in \Gamma(TM)$.

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$T \equiv 0$. $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

$$T(fX, Y) = f\nabla_X Y - Y(f)X - f\nabla_Y X - [fX, Y].$$

$$[fX, Y]h = fX(Y(h)) - Y(fX(h))$$
$$= f\cdot X(Y(h)) - Y(f)X(h) - fY(X(h)), \ \forall h$$

$$\Rightarrow [fX, Y] = f[X,Y] - Y(f)X$$

$$\Rightarrow T(fX, Y) = fT(X,Y). \qquad \square$$

Conclusion. ∇ affine connection s.t.

(a) ∇ compatible with g

(b) $T \equiv 0$ torsion-free

Definition

Levi-Civita connection

(Christoffel connection

Existence? Uniqueness?

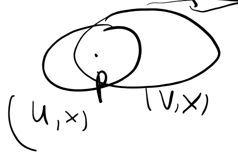
Define ~~as~~ a connection:

$\forall X, Y \in \Gamma(TM)$, $\nabla_X Y \in \Gamma(TM)$, s.t. $\forall p \in M$, $p \in U$, (U, x) chart.

n^3 $\left\{ {k \atop ij} \right\}$

$$\boxed{\nabla_X Y(p) := \underbrace{\left(X^i \frac{\partial Y^k}{\partial x^i} + X^i Y^j \Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}}}$$

well-defined?



independent of choices of charts.

transformation formula of Γ_{ij}^k $\Rightarrow$ Exercise

$$\boxed{\Gamma_{jk}^i(x) = \tilde{\Gamma}_{\eta\gamma}^{\alpha}(y(x))\frac{\partial y^\eta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial x^i}{\partial y^\alpha} + \frac{\partial^2 y^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^\alpha}.}$$

$\left\{ {i \atop jk} \right\}$ $\qquad \left\{ {\alpha \atop \eta\gamma} \right\}$ $\qquad\square$

休息到 2:55.

What is an affine connection?

$\nabla \leftrightarrow \left\{ {i \atop jk} \right\}$ n^3 C^∞-fct on each local chart (U, x)

~~s.t.~~ which have the "correct" transformation ~~behavio~~ behavior. $(X^1, \cdots, X^n)$

$$X = X^i \frac{\partial}{\partial x^i} \qquad \boxed{\frac{\partial X^i}{\partial x^k} + \Gamma_{hk}^i X^h}$$

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tensor

$$A^{i_1 \cdots i_r}_{\ \ j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

tensor

$$A^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}$$

$$\frac{\partial\left(A^{i_1\cdots i_r}_{j_1\cdots j_s}\right)}{\partial x^{u}} + ?\,A^{i_1\cdots i_r}_{j_1\cdots j_s}$$

Thm (The fundamental theorem of Rie. geometry)

On any Rie. mfld (M,g), $\exists!$ Levi-Civita connection.

(M,g) Rie mfld

- metric $\Rightarrow$ distance fct.
- metric $\Rightarrow$ Rie. measure
- metric $\Rightarrow$ Levi-Civita connection

Proof: Suppose ∇. Assume existence, ∇

$\forall\, X,Y\in\Gamma(TM)$ $\quad\langle\nabla_X Y, Z\rangle = g(\nabla_X Y, Z)$

$\forall\, Z\in\Gamma(TM)$

$$\langle\nabla_X Y, Z\rangle \overset{\nabla g\equiv 0}{=} X\langle Y,Z\rangle - \langle Y, \nabla_X Z\rangle \qquad \nabla_X Z-\nabla_Z X-[X,Z]=0$$

$$\overset{T\equiv 0}{=} X\langle Y,Z\rangle - \langle Y, \nabla_Z X + [X,Z]\rangle$$

$$= X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - \langle Y,\nabla_Z X\rangle$$

$$= X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - \left(Z\langle Y,X\rangle - \langle\nabla_Z Y, X\rangle\right)$$

$$= X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle\nabla_Y Z, X\rangle + \langle [Z,Y],X\rangle$$

$$= X\langle Y,Z\rangle - Z\langle Y,X\rangle - \langle Y,[X,Z]\rangle + \langle [Z,Y],X\rangle + Y\langle Z,X\rangle - \langle Z, \underbrace{\nabla_Y X}_{=\nabla_X Y + [Y,X]}\rangle$$

$$= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle [Z,Y],X\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle - \langle Z,\nabla_X Y\rangle$$

Koszul formula

$$\Rightarrow \boxed{2\langle\nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle} \quad (*)$$

$\forall\, X,Y,Z\in\Gamma(TM)$

$\Rightarrow$ uniqueness.

$\Rightarrow$ uniqueness.

Existence. define the connection via ($*$). $\square$

$X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$

geodesic. $\nabla_{\dot\gamma} \dot\gamma = 0$ $\frac{D\dot\gamma}{dt} = 0$ $\dot\gamma(t)$ a vector field along γ.

Lemma. $V(t), W(t)$ are two C^∞ vector fields along a curve c.

Then $\frac{d}{dt}\langle V(t), W(t)\rangle = \langle \frac{DV}{dt}, W(t)\rangle + \langle V(t), \frac{DW}{dt}\rangle$

Proof. (U, x) $c(t) = (c^1(t), \dots, c^n(t))$

$V(t) = V^i(t)\frac{\partial}{\partial x^i}(c(t))$, $W(t) = W^i(t)\frac{\partial}{\partial x^i}(c(t))$

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \frac{d}{dt}\left(V^i(t)W^j(t)\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \circ (c(t))\right)$$

$$= \frac{d}{dt}\left(V^i(t)W^j(t)\right)\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \circ c(t)$$

$$+ V^i(t)W^j(t) \cdot \underbrace{\frac{d}{dt}\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle \circ c(t)}$$

$$\| $$

$$dc\left(\frac{d}{dt}\right)\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$$

$$\|$$

$$\langle \nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle + \langle \frac{\partial}{\partial x^i}, \nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}\rangle$$

$$= \langle \dot V^i(t)\frac{\partial}{\partial x^i} + V^i(t)\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^i}, W^j(t)\frac{\partial}{\partial x^j}\rangle$$

$$+ \langle V^i(t)\frac{\partial}{\partial x^i}, \dot W^j(t)\frac{\partial}{\partial x^j} + W^j(t)\nabla_{dc(\frac{d}{dt})}\frac{\partial}{\partial x^j}\rangle$$

$$= \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle. \quad \square$$

Cor. geodesic eq: $\frac{D\dot\gamma}{dt} \equiv 0$.

Cor. (M, g, ∇), ∇ compatible with $g \iff$ any parallel transport is an isometry

Proof: $c : [a,b] \to M$ C^∞ curve

$P_{c,a,b} : T_{c(a)}M \to T_{c(b)}M$ $\langle\cdot,\cdot\rangle$ $\langle\cdot,\cdot\rangle$

linear isometry preserve $\langle\cdot,\cdot\rangle$ isometry

"$\Rightarrow$" $P_{c,a,t}(V_a) = V(t)$ $\quad V(b) := V_b$

$P_{c,a,t}(W_a) = W(t)$, $\quad W(b) := W_b$

$\Rightarrow V(b) = P_{c,a,b}(V_a)$, $W(b) = P_{c,a,b}(W_a)$

$\checkmark$ $\langle V(a), W(a)\rangle = \langle V_b, W_b\rangle$.

$$\frac{d}{dt}\langle V(t), W(t)\rangle = 0$$

$$\underset{\nabla g = 0}{=} \langle \overset{=0}{\frac{DV}{dt}}, W\rangle + \langle V, \overset{=0}{\frac{DW}{dt}}\rangle$$

$\Leftarrow$ If parallel transports are isometry.

? $\forall X, Y, Z \in \Gamma(TM)$,

$\underline{X\langle Y, Z\rangle}(p) \overset{?}{=} \underline{\langle \nabla_X Y, Z\rangle}(p) + \underline{\langle Y, \nabla_X Z\rangle}(p)$, $\forall p \in M$

Consider $X\underline{\langle Y, Z\rangle}(p) = X(p)\langle Y, Z\rangle$

Find a curve c. s.t. $c(0) = p$, $\dot{c}(0) = X(p)$.

$$X(p)\langle Y, Z\rangle = \frac{d}{dt}\Big|_{t=0}\langle Y(c(t)), Z(c(t))\rangle$$

T_pM. $\{E_1, \dots, E_n\}$ orthonormal basis $X(p)$ $\quad \langle E_i, E_j\rangle = \delta_{ij}$

$\underline{\{E_1(t), \dots, E_n(t)\}}$ $\underline{\text{orthonormal}}$ $c(0) = p$

$\{E_1(t), \ldots, E_n(t)\}$ orthonormal frame field. $c(0)=p$

$E_i(t) = P_{c,0,t}(E_i)$

$$X(p)\langle Y, Z\rangle = \frac{d}{dt}\Big|_{t=0} \langle Y^i(t) E_i(t), Z^j(t) E_j(t)\rangle$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\sum_{i=1}^n Y^i(t) Z^i(t)\right)$$

$$= \sum_{i=1}^n \left(\dot{Y}^i(t) Z^i(t) + Y^i(t)\dot{Z}^i(t)\right)$$

$$= \langle \dot{Y}^i(t) E_i(t), Z^j(t) E_j(t)\rangle + \langle Y^i(t) E_i(t), \dot{Z}^j(t) E_j(t)\rangle$$

$$= \left\langle \frac{DY}{dt}\Big|_{t=0}, Z\right\rangle + \left\langle Y, \frac{DZ}{dt}\Big|_{t=0}\right\rangle$$

$$= \langle \nabla_{X_p} Y, Z\rangle + \langle Y, \nabla_{X_p} Z\rangle \quad \square$$

geodesic eq. $\Leftrightarrow \frac{D\dot{\gamma}}{dt} \equiv 0$.

variation of energy functional.

下课.