· The first variation of energy functional.

(M, g, ∇) → Levi-Civita connection

∇ : · metric compatibility $\nabla_X\big(g(Y,Z)\big) = \cancel{(\nabla_X g)(Y,Z)} + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

· torsion-free

$T \equiv 0$

i.e. $\nabla_X g = 0$, $\forall X \in \Gamma(TM)$

more practical version:

Consider vector fields along a curve $c(t)$:

$Y(t)$, $Z(t)$

$$\frac{d}{dt}\langle Y(t), Z(t)\rangle = \frac{d}{dt} g(Y(t), Z(t))$$

$$= g\left(\frac{DY}{dt}(t), Z(t)\right) + g\left(Y(t), \frac{DZ}{dt}(t)\right).$$

Torsion-free: ~~$\forall X, Y, Z \in \Gamma(TM)$~~

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$, $\forall X, Y \in \Gamma(TM)$.

In local chart (U, x), $\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}}$

$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

Lemma: M, C^∞ mfld, ∇ torsion-free affine connection.

Let $S : \Omega \subset \mathbb{R}^2 \longrightarrow M$ be a C^∞ map

$\forall (x,y) \in \Omega$, assign $V(x,y) \in T_{S(x,y)}M$

Then $V(x,y)$ is a vector field along the map S.

$(U, \underline{x})$ $\quad V(x,y) = V^i(x,y)\frac{\partial}{\partial x^i}$

For convenience, we denote

$$dS\left(\frac{\partial}{\partial x}\right) := \frac{\partial S}{\partial x}, \quad dS\left(\frac{\partial}{\partial y}\right) =: \frac{\partial S}{\partial y}.$$

$\frac{D}{\partial x}$: covariant derivative along the curve $t \longmapsto S(t, y)$,

$\frac{D}{\partial y}$: covariant derivative along the curve $t \longmapsto S(x, t)$

$\frac{\partial}{\partial y}$: covariant ...

$t \longmapsto s(x, t)$

Then, we have

$$\frac{D}{\partial y}\frac{\partial s}{\partial x} = \frac{D}{\partial x}\frac{\partial s}{\partial y}$$

$x : U \longrightarrow \mathbb{R}^n$

$x^{-1} : \mathbb{R}^n \longrightarrow U$

Proof. Consider a local chart $(U, (x^1, \dots, x^n))$

$s : \mathbb{R}^2 \longrightarrow M$

$(x,y) \longmapsto s(x,y) := (s^1(x,y), \dots, s^n(x,y))$

$$\frac{\partial s}{\partial y} = ds\left(\frac{\partial}{\partial y}\right) = \frac{\partial s^i}{\partial y}(x,y)\frac{\partial}{\partial x^i}$$

$\nabla_X Y(p) = \nabla_{X(p)} Y$

$$\frac{\partial s}{\partial x} = ds\left(\frac{\partial}{\partial x}\right) = \frac{\partial s^i}{\partial x}(x,y)\frac{\partial}{\partial x^i}$$

$$\frac{D}{\partial x}\left(\frac{\partial s}{\partial y}\right) = \frac{\partial}{\partial x}(x,y)\frac{\partial s^i}{\partial x^i} + \frac{\partial s^i}{\partial y}(x,y)\nabla_{ds\left(\frac{\partial}{\partial x}\right)}\frac{\partial}{\partial x^i}$$

$ds\left(\frac{\partial}{\partial x}\right) = \frac{\partial s^j}{\partial x}\frac{\partial}{\partial x^j}$

$$= \boxed{\frac{\partial^2 s^i(x,y)}{\partial x \partial y}}\frac{\partial}{\partial x^i} + \frac{\partial s^i}{\partial y}\frac{\partial s^j}{\partial x}\boxed{\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}}$$

✓ ✓ ✓ ⟵ torsion-free

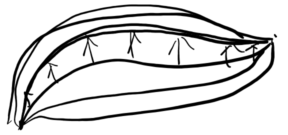
$$= \frac{D}{\partial y}\frac{\partial s}{\partial x}$$

□

C^∞ curve $c : [a,b] \to M$

$$E(c) = \frac{1}{2}\int_a^b \langle dc\left(\frac{d}{dt}\right), dc\left(\frac{d}{dt}\right)\rangle dt$$

$dc\left(\frac{d}{dt}\right) = c'(t)$



$c(t)$

$F(s,t)$ $\quad F(0,t) = c(t)$

$F : \mathbb{R}^2 \longrightarrow M$ $\quad C^\infty$

Definition. Let $c : [a,b] \to M$ C^∞ curve. $\varepsilon > 0$.

A variation of c is a C^∞ map:

$$F : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$$

with $F(t, 0) = c(t)$, $\forall t \in [a,b]$.

$\frac{\partial F}{\partial s} := dF\left(\frac{\partial}{\partial s}\right)$

$\frac{\partial F}{\partial t} := dF\left(\frac{\partial}{\partial t}\right)$

$F(a, s) = c(a)$, $F(b, s) = c(b)$ in Chap 2.

$V(t) := \frac{\partial F}{\partial s}(t, 0)$ is a vector field along $c(t)$.

Variational field

Thm (The First variation formula) Let $F: [a,b] \times (-\varepsilon, \varepsilon) \to M$ be a variation of $c(t)$, denote $E(c_s) := E(s)$, where $c_s(t) := F(s,t)$. Then we have

$$\boxed{\left.\frac{d}{ds}\right|_{s=0} E(s) = E'(0) = \langle V(b), c'(b)\rangle - \langle V(a), c'(a)\rangle - \int_a^b \langle V(t), \frac{Dc'(t)}{dt}\rangle dt.}$$

Remark: If $V(a) = V(b) = 0$, $E'(0) = 0$ for any F.

$$\Leftrightarrow \boxed{\frac{Dc'(t)}{dt} = 0}$$

Proof:

$$\left.\frac{d}{ds}\right|_{s=0} E(s) = \left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_a^b \langle c_s'(t), c_s'(t)\rangle dt$$

$$= \left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_a^b \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle dt$$

$$= \frac{1}{2}\int_a^b \underbrace{\frac{d}{ds}\langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle}\, dt \Big|_{s=0}$$

$$\overset{\text{metric compatibility}}{=} \int_a^b \langle \underbrace{\frac{D}{\partial s}\frac{\partial F}{\partial t}}, \frac{\partial F}{\partial t}\rangle dt \Big|_{s=0}$$

$$\overset{\text{torsion-free}}{=} \int_a^b \langle \frac{D}{\partial t}\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\rangle dt \Big|_{s=0} \qquad \frac{\partial F}{\partial t}(0,t) = c'(t)$$

$$\overset{\text{metric compatibility}}{=} \int_a^b \frac{d}{dt}\langle \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle - \langle \frac{\partial F}{\partial s}, \frac{D}{\partial t}\frac{\partial F}{\partial t}\rangle dt \Big|_{s=0}$$

$$= \int_a^b \frac{d}{dt}\langle V(t), c'(t)\rangle - \langle V(t), \frac{Dc'(t)}{dt}\rangle dt$$

$$= \langle V(b), c'(b)\rangle - \langle V(a), c'(a)\rangle - \int_a^b \langle V(t), \frac{Dc'(t)}{dt}\rangle dt \qquad \square$$

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Gauss Lemma.

normal coordinate. (U, x)

Rie. polar coordinate. (r, φ) $\quad \varphi = (\varphi^1, \cdots, \varphi^{n-1})$

$g = dr \otimes dr + g(r,\varphi)\, d\varphi \otimes d\varphi$

Rie. polar coordinate. (r, φ)

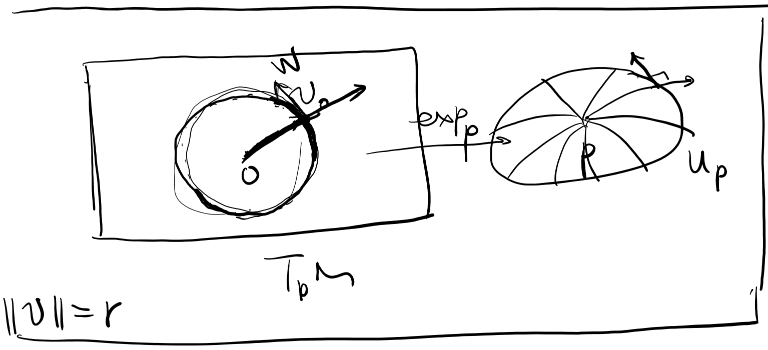
$$g = dr \otimes dr + g_{\varphi\varphi}(r,\varphi)\, d\varphi \otimes d\varphi$$

$$g_{r\varphi}(r,\varphi) \equiv 0 \qquad \cancel{g_{r\varphi}\, dr \otimes d\varphi}$$

Lemma: (Gauss' lemma) Let (U_p, x) be a normal neighborhood of p.

Then the radial geodesic from p are perpendicular to the hypersurfaces:

$$\{\exp_p(v) : v \in T_pM,\ \|v\| = r < \delta\}.$$



Precisely, let $v_0 \in T_pM$.

$\rho(t) = t v_0$, $t \in [0,1]$

a curve in T_pM

$\rho(0) = 0$, $\rho(1) = v_0$

~~Let~~ For any $W \in T_{v_0}(T_pM)$

s.t. $\langle W, \rho'(1) \rangle = 0$

Then $\left((d\exp_p)(v_0) : T_{v_0}(T_pM) \to T_{\exp_p(v_0)}M. \right)$

$$\langle (d\exp_p)(v_0)(W),\ (d\exp_p)(v_0)(\rho'(1)) \rangle = 0$$

$$\left(\langle W, v_0 \rangle = 0 \Rightarrow \langle (d\exp_p)(v_0)(W),\ (d\exp_p)(v_0)(v_0) \rangle = 0 \right)$$

Proof: We have a variation:

Consider a curve $v(s) : (-\varepsilon, \varepsilon) \to T_pM$

$v(0) = v_0$

s.t. $\dot{v}(0) = W$, $\|v(s)\| = \text{const}.$ $\Leftarrow$ Euclidean geometry

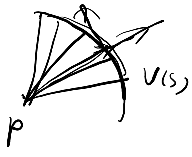
Consider the variation: $F : [0,1] \times (-\varepsilon, \varepsilon) \to M$

$(t, s) \mapsto \exp_p(t v(s))$

$c_s(t) := F(s,t)$

$$E(s) := \frac{1}{2} \int_0^1 \langle c_s'(t), c_s'(t) \rangle\, dt$$

$\equiv$ const.



$c_s(t) = \exp_p(t v(s))$

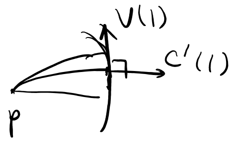
$\equiv$ const. $\quad c_s(t) = \exp_p(t\,\underline{\underline{v(s)}})$

$$= \frac{1}{2}\|v_s\|^2 \underline{\equiv} \text{ const.}$$

$$0 = E'(0) = \langle V(1), c'(1)\rangle - \langle \overset{=0}{V(0)}, c'(0)\rangle$$

$$- \int_0^1 \langle V(t), \underbrace{\frac{Dc'(t)}{dt}}_{=0} \rangle\, dt$$

$$= \underline{\langle V(1), c'(1)\rangle}$$



$$V(1) = \frac{\partial F}{\partial s}(t,0)\Big|_{t=1} = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(v(s))$$

$$= (d\exp_p)(v_0)(W)$$

$$c'(1) = (d\exp_p)(\overset{''\,v_0}{v_0})(\rho'(1))$$

$\square$

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§6. Covariant ~~deri~~ differentiation, Hessian and Laplacian

<u>covariant derivative</u> : for a (r,s)-tensor A

$$\nabla_X A \qquad \nabla_{\underbrace{fX+gY}_{\text{tensorial}}} A = f\nabla_X A + g\nabla_Y A.$$

<u>Therefore</u>, we can define a $(r, s+1)$-tensor, ∇A, as

$$\nabla A(\omega^1, \dots, \omega^r; X_1, \dots, X_s, X)$$

$$:= \nabla_X A(\omega^1, \dots, \omega^r; X_1, \dots, X_s) \quad \forall\, \omega^i \in \Gamma(T^*M),\ X_j \in \Gamma(TM).$$

We call ∇A the $\boxed{\text{covariant}}$ <u>differentiation</u> of A.

In a <u>local chart</u>, (U, x),

$$A = A^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

where $\underline{A^{i_1 \cdots i_r}_{j_1 \cdots j_s}} := A\left(dx^{i_1}, \dots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}}\right)$

where $A^{i_1\cdots i_r}_{j_1\cdots j_s} := A\left(dx^{i_1}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}\right)$

$$\frac{\partial}{\partial x^k} A^{i_1\cdots i_r}_{j_1\cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \cancel{\otimes} \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes dx^k$$

is not a tensor

$$\boxed{\nabla A = B^{i_1\cdots i_r}_{j_1\cdots j_s k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \otimes dx^k}$$

$$B^{i_1\cdots i_r}_{j_1\cdots j_s k} = \nabla A\left(dx^{i_1}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}, \frac{\partial}{\partial x^k}\right)$$

$$= \left(\nabla_{\frac{\partial}{\partial x^k}} A\right)\left(dx^{i_1}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}\right)$$

Comment. $A = Y \otimes \omega \qquad \nabla_X A = \nabla_X Y \otimes \omega + Y \otimes \nabla_X \omega$

$$\Rightarrow (\nabla_X A)(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s)$$

$$= X\left(A(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s)\right) - \sum_{i=1}^{r} A(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^r, Y_1, \ldots, Y_s)$$

$$- \sum_{j=1}^{r} A(\omega^1, \ldots, \omega^r, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_s)$$

Exercise. $(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y)$

$$B^{i_1\cdots i_r}_{j_1\cdots j_s k} = \frac{\partial}{\partial x^k}\left(A\left(dx^{i_1}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}\right)\right)$$

$$- \sum_{l=1}^{r} A\Big(dx^{i_1}, \ldots, \underbrace{\nabla_{\frac{\partial}{\partial x^k}} dx^{i_l}}_{-\Gamma^{i_l}_{kh} dx^h}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}\Big)$$

$$- \sum_{l=1}^{s} A\Big(dx^{i_1}, \ldots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \ldots, \underbrace{\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^{j_l}}}_{= \Gamma^h_{k j_l} \frac{\partial}{\partial x^h}}, \ldots, \frac{\partial}{\partial x^{j_s}}\Big)$$

$$\boxed{B^{i_1\cdots i_r}_{j_1\cdots j_s k} = \frac{\partial}{\partial x^k} A^{i_1\cdots i_r}_{j_1\cdots j_s} + \sum_{l=1}^{r} \Gamma^{i_l}_{kh} A^{i_1\cdots i_{l-1} h\, i_{l+1}\cdots i_r}_{j_1\cdots j_s} - \sum_{l=1}^{s} \Gamma^h_{k j_l} A^{i_1\cdots i_r}_{j_1\cdots j_{l-1} h\, j_{l+1}\cdots j_s}} \quad (*)$$

"covariant derivative" of $\{A^{i_1\cdots i_r}_{j\cdots}\}$

"Covariant derivative" of $\boxed{\left\{A^{i_1\cdots i_r}_{j_1\cdots j_s}\right\}_{i_1,\cdots,i_r,j_1,\cdots,j_s}}$

X vector field, $\{X^i\}_{i=1,\cdots,n}$

$$\boxed{\frac{\partial X^i}{\partial x^k}+\Gamma^i_{kh}X^h} \quad \text{is } (\nabla X)\left(\frac{\partial}{\partial x^k}\right)$$

1-form $\omega = f_i dx^i$ $\quad (f_1,\cdots,f_n)$

$$\boxed{\frac{\partial f_i}{\partial x^k}-\Gamma^h_{ki}f_h} \quad (\nabla\omega)\left(\frac{\partial}{\partial x^k}\right)$$

$$A^{i_1\cdots i_r}_{j_1\cdots j_s;k} := \nabla_k A^{i_1\cdots i_r}_{j_1\cdots j_s} =: B^{i_1\cdots i_r}_{j_1\cdots j_s k}$$

$g_{ij,k} = \frac{\partial}{\partial x^k}g_{ij}$ $\qquad$ $\partial_k A^{i_1\cdots i_r}_{j_1\cdots j_s}$, $\partial_k = \frac{\partial}{\partial x^k}$

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$$A, \quad \nabla A, \quad \nabla^2 A = \nabla(\nabla A), \quad \nabla^3 A, \quad \cdots$$

(r,s)-tensor, $(r,s+1)$-tensor, $(r,s+2)$-tensor, $(r,s+3)$-tensor

Warning! $\nabla^2 A(\omega^1,\cdots,\omega^r;X_1,\cdots,X_s,X,Y)$

$$\neq \nabla_Y\nabla_X A(\omega^1,\cdots,\omega^r;X_1,\cdots,X_s)$$

Lemma. For any (r,s)-tensor A, we have

- $\nabla^2 A(\cdots,X,Y) = (\nabla_Y(\nabla_X A))(\cdots) - (\nabla_{\nabla_Y X}A)(\cdots)$.

Proof. $\nabla(\nabla A)(\omega^1,\cdots,\omega^r;X_1,\cdots,X_s,X,Y)$

$$=(\nabla_Y(\nabla A))(\omega^1,\cdots,\omega^r;X_1,\cdots,X_s,X)$$

$$=Y\left((\nabla_X A)(\omega^1,\cdots,\omega^r;X_1,\cdots,X_s,\cancel{X})\right)$$

$$-1\Big(\ \overline{\qquad\qquad\qquad}$$

$$-\sum_{i=1}^{r}\nabla_X A(\omega^1,\dots,\nabla_Y\omega^i,\dots,\omega^r;X_1,\dots,X_s,\cancel{X})$$

$$-\sum_{j=1}^{s}\nabla_X A(\omega^1,\dots,\omega^r,X_1,\dots,\nabla_Y X_j,\dots,X_s,\cancel{X})\Big\}$$

$$-\nabla_{\nabla_Y X} A(\omega^1,\dots,\omega^r,X_1,\dots,X_s,\cancel{\nabla_Y X})$$

$$=\nabla_Y(\nabla_X A)\ \underline{(\omega^1,\dots,\omega^r;X_1,\dots,X_s)}-\nabla_{\nabla_Y X}A\,(\omega^1,\dots,\omega^r,\underline{X_1,\dots,X_s})\,.$$

$\square$

$f\in C^\infty(M)$, $(0,0)$-tensor $\qquad$ $\underline{\mathrm{grad} f}$ $=\cancel{\nabla f}$

∇f $\quad(0,1)$ tensor

$$\nabla f(X)=\nabla_X f=X(f)=df(X),\ \forall X\in\Gamma(TM)$$

$$\Rightarrow\quad \nabla f=df$$

$\nabla^2 f$ $\quad(0,2)$-tensor

$$\nabla^2 f(X,Y)=\nabla_Y(\nabla_X f)-\nabla_{\nabla_Y X}f$$

$$=\underline{Y(Xf)-\nabla_Y X(f)},\ \forall X,Y\in\Gamma(TM)$$

We call $\nabla^2 f$ the Hessian of f.

$$\mathrm{Hess}\, f:=\nabla^2 f.$$

<u>Lemma</u>: $\nabla^2 f(X,Y)-\nabla^2 f(Y,X)=T(X,Y)f$ $\quad\forall X,Y\in\Gamma(TM)$, $\forall f\in C^\infty(M)$

<u>Proof</u>: $\nabla^2 f(X,Y)-\nabla^2 f(Y,X)$

$$=Y(Xf)-\nabla_{\nabla_Y X}f-X(Yf)+\nabla_{\nabla_X Y}f$$

$$=[Y,X]f\ \underline{-(\nabla_Y X-\nabla_X Y)}f$$

$= [Y, X] f - \underline{(\nabla_Y X - \nabla_X Y)} f$

$= (\nabla_X Y - \nabla_Y X) f - [X, Y] f = T(X, Y) f$. □

torsion-free, $\nabla^2 f$ symmetric.

Remark $f : \mathbb{R}^n \to \mathbb{R}$. C^∞. Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)_{i,j=1,\cdots,n}$$

(U, x) $\boxed{\nabla^2 f \left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right)} = \frac{\partial^2 f}{\partial x^k \partial x^l} - \underbrace{\nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^k}}_{\Gamma^h_{lk} \frac{\partial}{\partial x^h}} f$

$= \boxed{\frac{\partial^2 f}{\partial x^k \partial x^l} - \Gamma^h_{lk} \frac{\partial f}{\partial x^h}}$ $\quad \nabla^2 f = \underline{\nabla^2 f \left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right) dx^k \otimes dx^l}$

f $(0,0)$-tensor. ∇f $(0,1)$ tensor.

$\underline{f_{;k}} = \frac{\partial f}{\partial x^k}$ $\quad \nabla f = \frac{\partial f}{\partial x^k} dx^k$

$\nabla^2 f$ $(0, 2)$ $\quad \underline{\nabla^2 f} = \nabla(\underline{\nabla f})$

$\underline{(f_{;k})}_{;l} = \frac{\partial}{\partial x^l}(f_{;k}) - f_{;h} \Gamma^h_{lk} = \boxed{\frac{\partial^2 f}{\partial x^l \partial x^k} - \frac{\partial f}{\partial x^h} \Gamma^h_{lk}}$

(U, x) Hessian matrix of f

$$\left((f_{;k})_{;l} \right) = \left(f_{;kl} \right)$$

$$(f_{,k})_{,l} = f_{,kl} = \frac{\partial^2 f}{\partial x^k \partial x^l}$$

Reformulate the divergence of a vector field.

$X \in \Gamma(TM)$, $\quad \mathrm{div}(X)$

$L_X \underset{\text{volume form}}{\Omega} = \underline{\mathrm{div}(X)}\, \Omega$

$\mathbb{R}^n \quad X = (X^1, \cdots, X^n)$

$\underline{\mathrm{div} X = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}}$

(U, x) $\quad \underline{\mathrm{div}(X) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} X^i)}$, $G = \det(g_{ij})$

Claim. in (U, x). $\quad \mathrm{div}(X) = \boxed{\sum_{i=1}^n (X^i)_{;i}}$

$X = X^i \frac{\partial}{\partial x^i}$

Claim. in (U,x) $\quad X = \underline{X^i} \frac{\partial}{\partial x^i}$ $\quad \operatorname{div}(X) = \boxed{\sum_{i=1} (X^i)_{;i}}$

$$= \sum_{i=1}^{n} \left(\frac{\partial X^i}{\partial x^i} + \sum_{h=1}^{n} \Gamma^i_{ih} X^h \right)$$

Claim $\sum_{i=1}^{n} \left(\frac{\partial X^i}{\partial x^i} + \sum_h \Gamma^i_{ih} X^h \right) = \sum_i \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} X^i)$

Proof: $\text{LHS} = \sum_i \frac{\partial X^i}{\partial x^i} + \underline{\sum_{i,h} \Gamma^i_{ih} X^h}$

$\text{RHS} = \underline{\sum_i \frac{1}{\sqrt{G}} \left(\frac{\partial}{\partial x^i} \sqrt{G}\right) \cdot X^i} + \sum_i \frac{\partial X^i}{\partial x^i}$

Remains to show $\sum_{i,h} \Gamma^h_{hi} X^i = \sum_i \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \sqrt{G} \cdot X^i$

$\forall X$

Enough to show

$$\boxed{\sum_{h=1}^{n} \Gamma^h_{hi} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \sqrt{G}}$$

$$\Gamma^h_{hi} = \frac{1}{2} g^{hk} \left(g_{ki,h} + g_{hk,i} - g_{hi,k} \right) \qquad g = (g_{ij})$$

$$= \frac{1}{2} \underline{g^{hk} g_{hk,i}} = \frac{1}{2} \underline{\operatorname{tr}\left(g^{-1} \cdot \frac{\partial}{\partial x^i} g\right)}$$

$$= \frac{1}{2} \frac{\partial}{\partial x^i} \ln \underbrace{\det g}_{G} = \frac{\partial}{\partial x^i} \ln \sqrt{G} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \sqrt{G} \quad \square$$

$A(t)$ matrix-valued fct. nonsingular.

$$\operatorname{tr}\left(A(t)^{-1} \frac{d}{dt} A(t)\right) = \frac{d}{dt} \ln\left(\det A(t)\right)$$

Globally. $X \in \Gamma(TM)$ $\qquad (X^i)_{;i}$

$$\nabla X \in \Gamma(\underline{TM \otimes T^*M})$$

$$\nabla X : \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$Y \longmapsto \nabla_Y X$$

$$\nabla X(p) : T_pM \longrightarrow T_pM$$

$$\underline{\operatorname{tr} \nabla X} = \operatorname{div}(X)$$

$$\underline{\nabla X} = \nabla \left(X^i \frac{\partial}{\partial x^i}\right) = \nabla_{\frac{\partial}{\partial x^k}} \left(X^i \frac{\partial}{\partial x^i}\right) \otimes dx^k$$

$$= \underbrace{\left(\frac{\partial X^i}{\partial x^k} + X^h \Gamma^i_{kh} \right)}_{X^i{}_{;k}} \frac{\partial}{\partial x^i} \otimes dx^k$$

$$\cancel{\nabla X\left(\frac{\partial}{\partial x^i}\right)} \quad \nabla X(p) : T_pM \longrightarrow T_pM$$

$$\cancel{\nabla X} \qquad \frac{\partial}{\partial x^i} \longmapsto$$

$$\nabla X\left(\frac{\partial}{\partial x^i}\right)(p) = \left(\frac{\partial X^k}{\partial x^i} + X^h \Gamma^k_{ih} \right) \frac{\partial}{\partial x^k}$$

matrix

$$\mathrm{tr}\left(\frac{\partial X^k}{\partial x^i} + X^h \Gamma^k_{ih} \right)_{i,k=1,\dots,n}$$

$$= \frac{\partial X^i}{\partial x^i} + X^h \Gamma^i_{ih} = \sum_{i=1}^n X^i{}_{;i}$$

$$\mathrm{div}(X) = \mathrm{tr}(\nabla X) = \sum_{i=1}^n X^i{}_{;i} \qquad \square$$

下课.