第十六讲

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$$\mathrm{div}(X) := \mathrm{tr}(\nabla X) = \sum_{i=1}^n X^i{}_{;i} = \sum_{i=1}^n \left(\frac{\partial X^i}{\partial x^i} + X^h \Gamma^i_{ih}\right)$$

$$= \sum_{i=1}^n \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G} X^i) \quad \overset{?}{=\!=} \quad L_X \Omega = \mathrm{div}(\Omega)$$

Rmk: (a) $\mathrm{tr}(\nabla X) = C(\nabla X)$

(b) normal coordinate around p: $g_{ij}(p) = \delta_{ij}$

$g_{ij,k}(p) = 0 \Rightarrow \Gamma^i_{jk}(p) = 0$

$$\Rightarrow \quad g^{ij}{}_{,k}(p) = 0 \quad (g^{ij} g_{jk} = \delta^i_k)$$

$$G_{,k}(p) = 0 \quad (\text{Exercise})$$

$\forall p \in M$ (*) $\displaystyle\sum_{i=1}^n \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G} X^i)(p) = \sum_{i=1}^n \left(\frac{\partial X^i}{\partial x^i} + X^h \Gamma^i_{ih}\right)(p)$

Choose (U, x) be a normal coordinate around p

$$A^{i_1 \cdots i_r}_{j_1 \cdots j_s ; k}(p) = \frac{\partial A^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial x^k}(p)$$

$$\sum_{i=1}^n \frac{\partial X^i}{\partial x^i}(p) = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}(p) \qquad \square$$

Reformulate the Laplace-Beltrami operator:

" trace of a $(0,2)$-tensor S ", $S \in \Gamma(T^*M \otimes T^*M)$

S can also be considered as the map

$$S: \Gamma(TM) \longrightarrow \Gamma(T^*M) \xrightarrow{\#} \Gamma(TM)$$

$$X \longmapsto S(X, \cdot) \longmapsto \# S(X, \cdot)$$

g metric tensor. a non-degenerate bilinear form on each T_pM, $\forall p \in M$

$\Rightarrow$ musical isomorphism between $\Gamma(TM)$ and $\Gamma(T^*M)$

$$\flat : \Gamma(TM) \longrightarrow \Gamma(T^*M)$$

"flat" $\quad X \longmapsto g(X, \cdot) := \flat(X)$

$$\flat(X)(Y) = g(X,Y)$$

$$\# : \Gamma(T^*M) \longrightarrow \Gamma(TM)$$

"sharp" $\quad \omega \longmapsto \#\omega$,

$$\forall Y \in \Gamma(TM), \quad g(\#\omega, Y) := \omega(Y)$$

Remark: In a local chart (U, x)

$$\flat(X)\left(\frac{\partial}{\partial x^k}\right) = \flat\left(X^i \frac{\partial}{\partial x^i}\right)\left(\frac{\partial}{\partial x^k}\right) = g\left(X^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = X^i g_{ik}$$

$$\Rightarrow \quad \flat(X) = X^i g_{ik}\, dx^k$$

$$X \longmapsto \flat(X)$$

$$X^i \qquad \sum_{i=1}^n X^i g_{ik}$$ lowering the indies

$$\#\omega = \#(\omega_i dx^i)$$

$$g\Big(\underbrace{\#(\omega_i dx^i)}_{\substack{\| \\ B^l \frac{\partial}{\partial x^l}}}, \frac{\partial}{\partial x^k}\Big) := \omega_i \underbrace{dx^i\left(\frac{\partial}{\partial x^k}\right)} = \omega_k$$

$$\asymp B^l \underbrace{g_{lk} g^{kp}}_{\delta_l^p} = \omega_k g^{kp}$$

$$\Rightarrow \quad B^p = \omega_k g^{kp}$$

$$\omega \longmapsto \#\omega$$

$$\omega_k \qquad \omega_k g^{kp}$$ raising the indies $\square$

Define the trace of a $(0,2)$-tensor S as

$$\operatorname{tr} S := \operatorname{tr}\left(X \longmapsto \#S(X, \cdot)\right).$$

In local charts, $S = S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) dx^i \otimes dx^j$

$$= \underbrace{S_{ij}}\, dx^i \otimes dx^j$$

$$X \longmapsto \flat S(X,\cdot) \longmapsto \# S(X,\cdot)$$

$$\frac{\partial}{\partial x^i} \longmapsto S_{ij}\, dx^j \longmapsto \underbrace{(g^{jk} S_{ij})} \frac{\partial}{\partial x^k}$$

$\Rightarrow$ matrix $\left(g^{jk} S_{ij} \right)_{i,k=1,\dots,n}$

$$\operatorname{tr} S = \operatorname{tr}\left(g^{jk} S_{ij} \right)_{i,k} = \sum_{i,j=1}^{n} g^{ji} S_{ij}$$

$$S = S_{ij}\, dx^i \otimes dx^j$$

$$\operatorname{tr} S = g^{ij} S_{ij} \qquad \square$$

$\forall f \in C^\infty(M)$, $(0,2)$-tensor $\operatorname{Hess} f := \nabla^2 f = \nabla(\overset{df}{\overset{\shortparallel}{\nabla f}})$
$= \nabla df$

<u>Lemma</u>: $\forall f \in C^\infty(M)$, we have

$$\underset{\operatorname{div}(\operatorname{grad} f)}{\overset{\shortparallel}{\Delta f}} = \operatorname{tr} \operatorname{Hess} f .$$

<u>Proof</u>: $g(\nabla_X \operatorname{grad} f, Y) \overset{\nabla g \equiv 0}{=} X\left(\underbrace{g(\operatorname{grad} f, Y)}\right) - g(\operatorname{grad} f, \nabla_X Y)$

$$= X(Y(f)) - \nabla_X Y(f)$$

$$= \operatorname{Hess} f(Y, X)$$

$$\overset{T \equiv 0}{=\!=} \operatorname{Hess} f(X, Y)$$

$$\Rightarrow \# \operatorname{Hess} f(X,\cdot) = \nabla_X \operatorname{grad} f$$

$$\Rightarrow \operatorname{tr} \operatorname{Hess} f = \operatorname{tr}\left(X \mapsto \# \operatorname{Hess} f(X,\cdot) \right)$$

$$= \operatorname{tr}\left(X \mapsto \nabla_X \operatorname{grad} f \right)$$

$$= \mathrm{tr}(\nabla \,\mathrm{grad} f)$$

$$= \mathrm{div}(\mathrm{grad} f) = \Delta f. \quad \square$$

<u>Remark</u>: $\mathbb{R}^n$, $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^{i2}}$ $\quad g_{ij} = \delta_{ij}$

(M, g) $\quad \Delta f = \mathrm{tr}\, \mathrm{Hess} f = g^{ij} \,\mathrm{Hess} f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

$$= g^{ij} \,\nabla^2 f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

$$= g^{ij} \,(f_{;i})_{;j} = g^{ij} \, f_{;ij}$$

$$\Delta f = g^{ij} f_{;ij} = g^{ij}\left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial f}{\partial x^h} \Gamma_{ji}^h\right)$$

$$f_{;i} = \frac{\partial f}{\partial x^i}$$

$$(f_{;i})_{;j} = \frac{\partial f_{;i}}{\partial x^j} - f_{;h} \Gamma_{ji}^h$$

$$\Delta f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij} \frac{\partial f}{\partial x^j}\right)$$

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$$= g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij}\right) \frac{\partial f}{\partial x^j}$$

<u>Show</u>: $\boxed{- \sum_{ij} g^{ij} \Gamma_{ji}^h = \sum_i \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ih}\right)}$

$$= \sum_i \underbrace{\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}(\sqrt{G})}\, g^{ih} + \sum_i \frac{\partial}{\partial x^i} g^{ih}$$

$$= \sum_{ik} \Gamma_{ki}^k g^{ih} + \sum_i g^{ih}{}_{,i}$$

<u>Levi-Civita</u>: $\nabla g = 0$

$$g_{ij;k} = \boxed{0 = g_{ij,k} - g_{hj}\Gamma_{ki}^h - g_{ih}\Gamma_{kj}^h}$$

$$g_{ij;k} = \boxed{0 = g_{ij,k} - g_{hj}\Gamma^h_{ki} - g_{ih}\Gamma^h_{kj}}$$

Claim $$g^{ij}{}_{;k} = \boxed{0 = g^{ij}{}_{,k} + g^{hj}\Gamma^i_{kh} + g^{ih}\Gamma^j_{kh}}$$

In local charts.

$$A^{i_1\cdots i_r}_{j_1\cdots j_s\,;k}$$

$$\Big(\underbrace{\phi^{i_1\cdots i_r}_{j_1\cdots j_s}\psi^{h_1\cdots h_p}_{k_1\cdots k_q}}_{C(\phi\otimes\psi)}\Big)_{;k} = \left(\phi^{i_1\cdots i_r}_{j_1\cdots j_s}\right)_{;k}\psi^{h_1\cdots h_p}_{k_1\cdots k_q} + \phi^{i_1\cdots i_r}_{j_1\cdots j_s}\psi^{h_1\cdots h_p}_{k_1\cdots k_q;k}$$

Proof: $\nabla(\phi\otimes\psi) = \nabla\phi\otimes\psi + \phi\otimes\nabla\psi$ $\square$

$$\Big(\underbrace{\sum_{i_r=1}^{n}\phi^{i_1\cdots i_r}_{j_1\cdots j_s}\psi^{h_1\cdots h_p}_{i_r k_2\cdots k_q}}_{C(\phi\otimes\psi)}\Big)_{;k} = \sum_{i_r}\phi^{i_1\cdots i_r}_{j_1\cdots j_s;k}\psi^{h_1\cdots h_p}_{i_r k_2\cdots k_q} + \sum_{i_r}\phi^{i_1\cdots i_r}_{j_1\cdots j_s}\psi^{h_1\cdots h_p}_{i_r k_2\cdots k_q;k}$$

$$g_{ij;k} = 0 \Rightarrow g^{ij}{}_{;k} = 0,\ \forall k$$

$$(g_{ij}g^{jk})_{;p} = (\delta_i^k)_{;p} = \underbrace{\frac{\partial\delta_i^k}{\partial x^p}}_{=0} + \delta_i^h\Gamma^k_{ph} - \delta^k_h\Gamma^h_{pi}$$
$$= \Gamma^k_{pi} - \Gamma^k_{pi} = 0$$

$$\| $$

$$\underbrace{g_{ij;p}}_{0} g^{jk} + g_{ij}g^{jk}{}_{;p}$$

$$\Rightarrow g_{ij}g^{jk}{}_{,p} = 0 \Rightarrow g^{jk}{}_{,p} = 0,\ \forall p \quad \square$$

§7 Ricci identity.

$$A \in \Gamma(\otimes^{r,s}TM)$$

$$\nabla^2 A(\cdots, X, Y) = \nabla_Y\nabla_X A(\cdots) - \nabla_{\nabla_Y X}A(\cdots)$$

$$\nabla^2 A(\cdots, X, Y) - \nabla^2 A(\cdots, Y, X)$$

$$= \left(\nabla_Y \nabla_X A - \nabla_{\nabla_Y X} A - \nabla_X \nabla_Y A + \nabla_{\nabla_X Y} A \right)(\cdots)$$

$$= - \left(\nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{\underbrace{\nabla_X Y - \nabla_Y X}_{= [X,Y]}} A \right)(\cdots) \quad T \equiv 0$$

$$= - \left(\nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A \right)(\cdots) \quad (\star)$$

Definition: $\forall\, X, Y \in \Gamma(TM)$, $\forall\, A \in \Gamma(\otimes^{r,s} TM)$, define

$$R(X,Y)A := \nabla_X(\nabla_Y A) - \nabla_Y(\nabla_X A) - \nabla_{[X,Y]} A$$

Theorem (Ricci Identity). $\forall\, X, Y \in \Gamma(TM)$, $\forall A \in \Gamma(\otimes^{r,s} TM)$ we have

$$\nabla^2 A(\cdots, X, Y) - \nabla^2 A(\cdots, Y, X)$$

$$= - R(X,Y)A(\cdots). \qquad \square$$

Remark: (a) $R(X,Y)A = -R(Y,X)A$

(b) $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(\otimes^{r,s} TM) \to \Gamma(\otimes^{r,s} TM)$

$$(X, Y, A) \longmapsto R(X,Y)A$$

is linear over C^∞ fcts w.r.t. all of its arguments

Proof: $R(fX_1 + gX_2, Y)A = fR(X_1, Y)A + gR(X_2, Y)A$

$$\boxed{R(X,Y)(fA) \overset{?}{=} fR(X,Y)A}$$

$R(X,Y)(A_1 + A_2) = R(X,Y)A_1 + R(X,Y)A_2$ ✓

$\nabla^2(fA)(\cdots, X, Y) \neq f\nabla^2 A(\cdots, X, Y)$!!

Exercise. $\square$

<u>locality</u>: $\underline{R(X,Y)A}(p)$ is only dependent on $X(p)$, $Y(p)$, $A(p)$.

Only need consider on (U,x)

$$R\left(X^i\frac{\partial}{\partial x^i}, Y^j\frac{\partial}{\partial x^j}\right)\left(A^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}\right)_p$$

$$= X^i(p)Y^j(p)A^{i_1\cdots i_r}_{j_1\cdots j_s}(p)\,R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}(p)$$

Levi-Civita connection

(c) <u>Particular case</u>: $(0,0)$-tensor f, $f\in C^\infty(M)$

$$\underline{R(X,Y)f} = \underline{\nabla^2 f(Y,X)} - \nabla^2 f(X,Y)$$

$$= -T(X,Y)f \overset{T=0}{=} 0$$

In (U,x) $(f_{;k})_{;l} = (f_{;l})_{;k}$

$$f_{;kl} = f_{;lk}$$

(d) <u>Particular case</u>: $(1,0)$-tensor. $Z\in\Gamma(TM)$

$$R(X,Y)Z = \nabla^2 Z(Y,X) - \nabla^2 Z(X,Y)$$

$$= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

$$R: \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(TM)$$

$$\underline{(X,Y,Z)} \longmapsto \underline{R(X,Y)Z},$$

R $(1,3)$-tensor in the sense.

$$R(\omega, X,Y,Z) := \omega(R(X,Y)Z)$$

We call R is a <u>curvature tensor</u>.

In local chart:

Notation. $R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} := \boxed{R^k{}_{lij}}\frac{\partial}{\partial x^k}$

Notation: $\underline{R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}} := \boxed{R^k{}_{lij}}\frac{\partial}{\partial x^k}$

$$\partial_i = \frac{\partial}{\partial x^i} \qquad R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = \nabla_{\partial_i}\underline{\nabla_{\partial_j}\partial_l} - \nabla_{\partial_j}\nabla_{\partial_i}\partial_l$$

$$= \nabla_{\partial_i}(\Gamma^\gamma_{jl}\partial_\gamma) - \nabla_{\partial_j}(\Gamma^\mu_{il}\partial_\mu)$$

$$= \partial_i(\Gamma^\gamma_{jl})\partial_\gamma + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma}\partial_k - \partial_j\Gamma^\mu_{il}\partial_\mu - \Gamma^\mu_{il}\Gamma^k_{j\mu}\partial_k$$

$$= \left(\partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma} - \Gamma^\mu_{il}\Gamma^k_{j\mu}\right)\partial_k$$

$$\Rightarrow \boxed{R^k{}_{lij}(p) = \frac{\partial\Gamma^k_{jl}}{\partial x^i} - \frac{\partial\Gamma^k_{il}}{\partial x^j} + \Gamma^\gamma_{jl}\Gamma^k_{i\gamma} - \Gamma^\mu_{il}\Gamma^k_{j\mu}}$$

Theorem (Ricci identity: local version)

Let $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ be the components of a (r,s)-tensor in (U,x).

Then

$$\left(A^{i_1\cdots i_r}_{j_1\cdots j_s}\right)_{;kl} - \left(A^{i_1\cdots i_r}_{j_1\cdots j_s}\right)_{;lk}$$

$$= \sum_{\alpha=1}^{s} A^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1}hj_{\alpha+1}\cdots j_s} R^h{}_{j_\alpha kl} - \sum_{\beta=1}^{r} A^{i_1\cdots i_{\beta-1}hi_{\beta+1}\cdots i_r}_{j_1\cdots j_s} R^{i_\beta}{}_{hkl}$$

□

Proof: This is a tensor identity. For any $p\in M$, take a normal coordinate of p. Then

$$\left(A^{i_1\cdots i_r}_{j_1\cdots j_s}\right)_{;kl}(p) = \frac{\partial}{\partial x^l}\left(A^{i_1\cdots i_r}_{j_1\cdots j_s;k}\right)(p)$$

$$= \partial_l\left(\partial_k A^{i_1\cdots i_r}_{j_1\cdots j_s} + \sum_{\beta=1}^{r} A^{i_1\cdots i_{\beta-1}hi_{\beta+1}\cdots i_r}_{j_1\cdots j_s}\Gamma^{i_\beta}_{kh} - \sum_{\alpha=1}^{s} A^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1}hj_{\alpha+1}\cdots j_s}\Gamma^h_{kj_\alpha}\right)(p)$$

$$= \partial^2_{l,k} A^{i_1\cdots i_r}_{j_1\cdots j_s}(p) + \sum_{\beta=1}^{r} A^{i_1\cdots i_{\beta-1} h\, i_{\beta+1}\cdots i_r}_{j_1\cdots j_s} \partial_l \Gamma^{i_\beta}_{kh}(p) - \sum_{\alpha=1}^{r} A^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1} h\, j_{\alpha+1}\cdots j_s} \partial_l \Gamma^{h}_{k j_\alpha}(p)$$

Therefore,

$$LHS(p) = \sum_{\beta=1}^{r} A^{i_1\cdots i_{\beta-1} h\, i_{\beta+1}\cdots i_r}_{j_1\cdots j_s} \underbrace{\left(\partial_l \Gamma^{i_\beta}_{kh} - \partial_k \Gamma^{i_\beta}_{lh}\right)(p)}_{= R^{i_\beta}{}_{hlk}(p)}$$

$$- \sum_{\alpha=1}^{s} A^{i_1\cdots i_r}_{j_1\cdots j_{\alpha-1} h\, j_{\alpha+1}\cdots j_s} \underbrace{\left(\partial_l \Gamma^{h}_{k j_\alpha} - \partial_k \Gamma^{h}_{l j_\alpha}\right)}_{= R^{h}{}_{j_\alpha l k}}(p)$$

$$= RHS(p) \quad \text{by} \quad R^{i_\beta}{}_{hlk} = -R^{i_\beta}{}_{hkl}, \quad R^{h}{}_{j_\alpha l k} = -R^{h}{}_{j_\alpha k l}. \ \square$$