第十八讲

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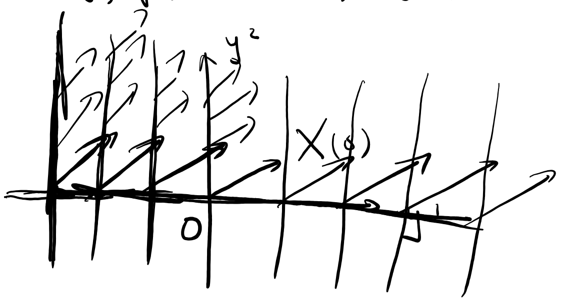
Remark $s: (y^1, y^2) \longmapsto (y^1, y^2, 0, \dots, 0) \in \mathbb{R}^n$

$X(0) \in T_0\mathbb{R}^n$



parallel transport $\rightsquigarrow$ $\underline{\underline{X}}$ vector field along s.

X is a $\underline{C^\infty}$ vector field along s

$$R\left(\frac{\partial s}{\partial y^1}, \frac{\partial s}{\partial y^2}\right) X = \frac{D}{\partial y^1}\frac{D}{\partial y^2}X - \frac{D}{\partial y^2}\frac{D}{\partial y^1}X$$

$X = X^i \frac{\partial}{\partial x^i}$ is parallel along c

$$\begin{cases} \left(\dfrac{dX^k}{dt} + X^i \dfrac{\partial c^j}{\partial t}\Gamma^k_{ij}\right) = 0, \quad k = 1, 2, \dots, n \\ X^k(0) = X^k_0 \end{cases}$$

$\rightsquigarrow$ X is C^∞ vector field along s

$$X = \underline{X^i(y^1, y^2)} \frac{\partial}{\partial y^i}$$

Curvature tensor $R: \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

$(X, Y, Z) \longmapsto R(X, Y)Z.$

Riemannian curvature tensor

Second Variation

Let $\gamma: [a, b] \to M$ be a geodesic, $|\dot\gamma(t)| = 1$

(a normal geodesic).

2-parameter variation F of γ, $\subset \mathbb{R}^3$

$$F: [a, b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \longrightarrow M$$

$$(t,\ \underline{v},\ \underline{w}) \longmapsto F(t, \underline{v}, \underline{w})$$

C^∞ map such that

$$F(t,0,0) = \gamma(t).$$

<u>Variational fields</u>:

$$V(t) := \frac{\partial F}{\partial v}(t,0,0), \quad W(t) := \frac{\partial F}{\partial w}(t,0,0)$$

Consider energy functional

$$E(v,w) = \frac{1}{2}\int_a^b \left\langle \frac{\partial F}{\partial t}(t,v,w), \frac{\partial F}{\partial t}(t,v,w) \right\rangle dt$$

$$\frac{\partial}{\partial v} E(v,w) = \frac{1}{2}\int_a^b \frac{\partial}{\partial v} \underbrace{\left\langle \frac{\partial F}{\partial t}(t,v,w), \frac{\partial F}{\partial t}(t,v,w) \right\rangle}\, dt$$

$$\underset{=}{\overset{\nabla g \equiv 0}{}} \int_a^b \left\langle \underline{\frac{D}{\partial v}\frac{\partial F}{\partial t}}, \frac{\partial F}{\partial t} \right\rangle dt$$

$$\frac{\partial^2}{\partial w \partial v} E(v,w) = \frac{\partial}{\partial w}\left(\int_a^b \left\langle \frac{D}{\partial v}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle dt \right)$$

$$= \int_a^b \frac{\partial}{\partial w} \left\langle \underline{\frac{D}{\partial v}\frac{\partial F}{\partial t}}, \frac{\partial F}{\partial t} \right\rangle dt$$

$$\underset{=}{\overset{\nabla g \equiv 0}{}} \int_a^b \left(\left\langle \frac{D}{\partial w}\frac{D}{\partial v}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \underline{\frac{D}{\partial v}\frac{\partial F}{\partial t}}, \underline{\frac{D}{\partial w}\frac{\partial F}{\partial t}} \right\rangle \right) dt$$

$$\underset{=}{\overset{T \equiv 0}{}} \int_a^b \left(\left\langle \underline{\frac{D}{\partial w}\frac{D}{\partial v}\frac{\partial F}{\partial t}}, \underline{\frac{\partial F}{\partial t}} \right\rangle + \left\langle \frac{D}{\partial t}\frac{\partial F}{\partial v}, \frac{D}{\partial t}\frac{\partial F}{\partial w} \right\rangle \right) dt$$

$$\boxed{\left.\frac{\partial^2}{\partial w \partial v}\right|_{v=w=0} E(v,w) = \int_a^b \left\langle \frac{D}{\partial w}\underline{\frac{D}{\partial t}\frac{\partial F}{\partial v}}, \frac{\partial F}{\partial t} \right\rangle\Big|_{v=w=0} dt + \int_a^b \left\langle \frac{D}{\partial t}V(t), \frac{D}{\partial t}W(t) \right\rangle dt}$$

<u>Lemma</u>: curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

$$R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial w}\right)\frac{\partial F}{\partial t} = \frac{D}{\partial v}\frac{D}{\partial w}\frac{\partial F}{\partial t} - \frac{D}{\partial w}\frac{D}{\partial v}\frac{\partial F}{\partial t}.$$

$$R\left(\frac{\partial F}{\partial v}, \quad \right)\frac{}{\partial t} = \frac{}{\partial v}\frac{}{\partial w}\frac{}{\partial t} - \frac{}{\partial w}\frac{}{\partial v}\frac{\partial F}{\partial t}.$$

$$\boxed{R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial v} = \frac{D}{\partial w}\frac{D}{\partial t}\frac{\partial F}{\partial v} - \frac{D}{\partial t}\frac{D}{\partial w}\frac{\partial F}{\partial v}}$$

$$\frac{\partial^2}{\partial w \partial v}\Big|_{v=w=0} E(v,w) = \int_a^b \left\langle \frac{DV}{\partial t}, \frac{DW}{\partial t}\right\rangle dt + \int_a^b \Big\langle \underbrace{R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t}\right)}_{W(t)\ \ \dot\gamma(t)}\underbrace{\frac{\partial F}{\partial v}}_{V(t)}, \underbrace{\frac{\partial F}{\partial t}}_{\dot\gamma(t)}\Big\rangle dt\Big|_{v=w=0}$$

$$+ \underbrace{\int_a^b \left\langle \frac{D}{\partial t}\frac{D}{\partial w}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle dt\Big|_{v=w=0}}_{\int_a^b \frac{\partial}{\partial t}\left\langle \frac{D}{\partial w}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle - \left\langle \frac{D}{\partial w}\frac{\partial F}{\partial v}, \cancel{\frac{D}{\partial t}\frac{\partial F}{\partial t}}\right\rangle dt\Big|_{v=w=0}}$$

$$\frac{D}{\partial t}\frac{\partial F}{\partial t} = 0$$

$$/\!/ \quad \frac{D}{\partial t}\dot\gamma(t) = 0 \Leftarrow \frac{\partial F}{\partial t}(t,0,0) = \dot\gamma(t)$$

$$\left\langle \frac{D}{\partial w}\frac{\partial F}{\partial v}(t,0,0), \dot\gamma(t)\right\rangle\Big|_{t=a}^{t=b}$$

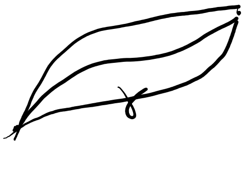
Rie. curvature tensor

SVF

$$\boxed{\begin{aligned}\frac{\partial^2}{\partial w \partial v}\Big|_{v=w=0} E(v,w) &= \int_a^b \left\langle \frac{DV}{\partial t}, \frac{DW}{\partial t}\right\rangle + \boxed{\langle R(W(t), \dot\gamma(t))V(t), \dot\gamma(t)\rangle} dt \\ &+ \left\langle \frac{D}{\partial w}\frac{\partial F}{\partial v}(t,0,0), \dot\gamma(t)\right\rangle\Big|_{t=a}^{t=b}\end{aligned}}$$

$$\underline{g_{mk}R^k{}_{lij}} = \underline{R_{mlij}}$$

$$F : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow M$$
$$(t, v) \longmapsto F(t,v)$$



$$\frac{\partial^2}{\partial v^2}E(v)\Big|_{v=0} = E''(0)$$
$$= \int_a^b \left\langle \frac{DV}{dt}, \frac{DV}{dt}\right\rangle + \langle R(V(t), \dot\gamma(t))V(t), \dot\gamma(t)\rangle dt$$
$$+ \left\langle \frac{D}{\partial v}\frac{\partial F}{\partial v}(t,0), \dot\gamma(t)\right\rangle\Big|_{t=a}^{t=b}$$

$$\begin{cases} F(a,v) = F(a,0), \ \forall v \in (-\varepsilon, \varepsilon) \\ F(b,v) = F(b,0), \ \forall v \in (-\varepsilon, \varepsilon)\end{cases}$$ fixed two endpts

$$\frac{D}{\ }\frac{\partial F}{\ }(t,0) \qquad \frac{\partial F}{\partial v}(a,v) = 0 \qquad \frac{\partial F}{\partial v}(b,v) = 0$$

$$\frac{D}{\partial v}\frac{\partial F}{\partial v}(t,0) \qquad \frac{\partial F}{\partial v}(a,v)=0 \qquad \frac{\partial F}{\partial v}(b,v)=0$$

$$E''(0)=\int_a^b \Big(\underbrace{\boxed{\Big\langle \frac{DV}{\partial t}, \frac{DV}{\partial t}\Big\rangle}}_{\geq 0} + \underbrace{\boxed{\langle R(V(t),\dot\gamma(t))V(t),\dot\gamma(t)\rangle}}_{\geq 0}\Big) dt$$

Remark: $\dot\gamma(t) = T(t)$ $\qquad \boxed{\frac{DV}{\partial t} = \nabla_T V}$

$$E''(0)=\int_a^b \{\langle \nabla_T V, \nabla_T V\rangle + \boxed{\langle R(V,T)V,T\rangle}\} dt$$

$$\boxed{I(V,V) = \int_a^b \{\langle \nabla_T V, \nabla_T V\rangle + \langle R(V,T)V,T\rangle\} dt}$$

$$\boxed{I(V,W) = \int_a^b \langle \nabla_T V, \nabla_T W\rangle + \langle R(W,T)V,T\rangle \, dt}$$

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Symmetries of the Rie. curvature tensor.

$\langle R(X,Y)Z, W\rangle$: $\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \to C^\infty(M)$

$(0,4)$-tensor

Notation: $R(W,Z,X,Y) := \langle R(X,Y)Z, W\rangle$

$\{R(W,Z,X,Y), \ X,Y,Z,W\in\Gamma(TM)\}$

$\boxed{\{R(X,Y,X,Y), \ X,Y\in\Gamma(TM)\}}$

$\parallel$

$\langle R(X,Y)Y,X\rangle$

In (U,x)

$$R_{klij} = R\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \Big\langle R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\Big\rangle$$

$$= \langle R^m \quad \partial \quad \partial \rangle = R^m \quad g$$

$$= \langle R^m{}_{lij} \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^k} \rangle = R^m{}_{lij}\, g_{mk}$$

$$\Rightarrow \quad R_{klij} = g_{km} R^m{}_{lij}$$

$$= g_{km} \left(\frac{\partial \Gamma^m_{jl}}{\partial x^i} - \frac{\partial \Gamma^m_{il}}{\partial x^j} + \Gamma^r_{jl} \Gamma^m_{ir} - \Gamma^r_{il} \Gamma^m_{jr} \right)$$

$$g_{km} \frac{\partial \Gamma^m_{jl}}{\partial x^i} = \frac{\partial}{\partial x^i} \left(\boxed{g_{km} \Gamma^m_{jl}} \right) - \Gamma^m_{jl}\, \boxed{g_{km,i}}$$

$$\nabla g = 0 \quad g_{km;i} = 0 \overset{\text{Ricci}}{=} g_{km,i} - g_{hm} \Gamma^h_{ik} - g_{kh} \Gamma^h_{im}$$

$$\boxed{R_{klij} = \frac{1}{2} \left(\underset{①}{\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}} - \underset{②}{\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}} - \underset{③}{\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}} + \underset{④}{\frac{\partial^2 g_{il}}{\partial x^j \partial x^k}} \right) + g_{mp} \left(\Gamma^m_{il} \Gamma^p_{jk} - \Gamma^m_{jl} \Gamma^p_{ik} \right)}$$

$\boxed{\text{Exercise}}$

Symmetries: $R_{klij} \underset{\circ}{=} -R_{klji} = -R_{lkij} \underset{\circ}{=} R_{ijkl}$

Proposition:

(1) $\langle R(X,Y)Z, W \rangle = -\langle R(Y,X)Z, W \rangle$

i.e. $R(W,Z,X,Y) = -R(W,Z,Y,X)$.

local: $R_{klij} = -R_{klji}$

(2) $\langle R(X,Y)W, Z \rangle = -\langle R(X,Y)Z, W \rangle$

i.e. $R(Z,W,X,Y) = -R(W,Z,X,Y)$

$R_{klij} = -R_{lkij}$

(3) $\langle R(X,Y)Z, W \rangle + \langle R(Y,Z)X, W \rangle + \langle R(Z,X)Y, W \rangle = 0$

i.e. $R(W,Z,X,Y) + R(W,X,Y,Z) + R(W,Y,Z,X) = 0$

$$\therefore\ R(W,Z,X,Y)+R(W,X,Y,Z)+R(W,Y,Z,X)=0$$

local $R_{klij}+R_{kijl}+R_{kjli}=0$

(4) $\langle R(X,Y)Z,W\rangle=\langle R(Z,W)X,Y\rangle$

$\boxed{\text{local } R_{ijkl}=R_{klij}}$

i.e. $R(W,Z,X,Y)=R(Y,X,Z,W)=R(X,Y,W,Z)$

(5) $\underbrace{\nabla R(W,Z,X,Y,V)}_{\|}+\nabla R(W,Z,Y,V,X)+\nabla R(W,Z,V,X,Y)=0$

$(\nabla_V R)(W,Z,X,Y)=\langle \nabla_V R(X,Y)Z,W\rangle$

local. $R_{klij;h}+R_{kljh;i}+R_{klhi;j}=0$

Proof: (2) $\langle R(X,Y)Z,W\rangle=-\langle R(X,Y)W,Z\rangle$

~~(4)~~ $\boxed{\langle R(X,Y)Z,W\rangle=-\langle Z,R(X,Y)W\rangle}$

$R(X,Y):\Gamma(TM)\to\Gamma(TM)$

$R(X,Y)=\nabla_X\nabla_Y\cdot-\nabla_Y\nabla_X\cdot-\nabla_{[X,Y]}$ $\boxed{\text{Excercise}}$

$\langle\nabla_X\nabla_Y Z,W\rangle\neq\langle Z,\nabla_X\nabla_Y W\rangle$

(4) $\langle R(X,Y)Z,W\rangle=\langle R(X,Y)W,Z\rangle$ (1),(2),(3)

$\langle R(X,Y)Z,W\rangle\overset{(3)}{=}-\langle R(Y,Z)X,W\rangle-\langle R(Z,X)Y,W\rangle$

$\langle R(X,Y)Z,W\rangle\overset{(2)}{=}-\langle R(X,Y)W,Z\rangle$

$\overset{(3)}{=}\langle R(Y,W)X,Z\rangle+\langle R(W,X)Y,Z\rangle$

$\overset{(2)}{=}-\langle R(Y,W)Z,X\rangle-\langle R(W,X)Z,Y\rangle$

$\|$ (1),(2)

$\langle R(Y,Z)W,X\rangle+\langle R(W,Y)Z,X\rangle$

$\|$ (3)

Excercise

Exercise

$$\overset{(3)}{=} -\langle R(Z,W)Y, X\rangle$$

$$\overset{(2)}{=} \langle R(Z,W)X, Y\rangle \ne \langle R(Z,W)X, Y\rangle 。$$

下课