# 第二讲

2020年2月27日 14:00

Riemann manifold $(M^n, g)$

$M^n$: n-dim $C^\infty$ manifold

$\forall p \in M$, $T_pM$ becomes an inner product

Whitney $M^n \longrightarrow (\mathbb{R}^{2n+1}, g_0)$

Given $(M^n, g) \longrightarrow (\mathbb{R}^N, g_0)$ isometrically embedding

Nash

**The metric structure**: distance function

$M$, $\quad d: M \times M \longrightarrow \mathbb{R}$ satisfying

(i) $d(p,q) \geq 0$, and $d(p,q) = 0 \Leftrightarrow p = q$

(ii) $d(p,q) = d(q,p)$,

(iii) $d(p,q) \leq d(p,r) + d(r,q)$ $\quad \cdot r \quad \dot{p} \quad \cdot q$

$(M, d)$ metric space

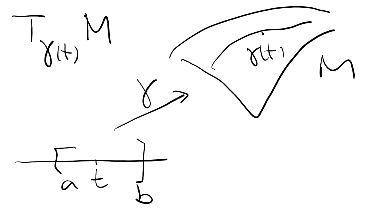
Fréchet 1906 neighborhood, limit, continuous

$\mathbb{R}^n$

Curves in $(M^n, g)$

$\gamma: [a,b] \longrightarrow M^n$ $\quad C^\infty$ curve $\gamma = \gamma(t)$

$$\dot{\gamma}(t) := d\gamma\left(\frac{d}{dt}\right) \in T_{\gamma(t)}M$$




$$\text{Length}(\gamma) := \int_a^b |\dot{\gamma}(t)|\, dt$$

$$= \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}}\, dt$$

**Lemma**: The Length$(\gamma)$ does not depend on the choices of parametrization.

**Proof**: $\gamma = \gamma(t)$, $t \in [a,b]$, $\gamma = \gamma_1(t_1)$, $t_1 \in [c,d]$

$a \leq b$, $c \leq d$

$$\text{Length}(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt \;\; = \;\; \int_c^d \sqrt{\langle \dot{\gamma}_1(t_1), \dot{\gamma}_1(t_1) \rangle}\, dt_1 = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \underbrace{\left|\frac{dt}{dt_1}\right| \frac{dt_1}{dt}}_{=1} dt$$

$t_1 = t_1(t) \; : [a,b] \rightarrow [c,d]$

$t = t(t_1) : [c, d] \to [a, b]$

$f: M \to \mathbb{R}$ $\quad \dot{\gamma}(t) \quad ? \quad \dot{\gamma}_1(t_1)$

$$\dot{\gamma}(t) f = d\gamma\left(\frac{d}{dt}\right) f = \frac{d}{dt}\left(f \circ (\gamma(t))\right)$$

$$\dot{\gamma}_1(t_1) f = d\gamma_1\left(\frac{d}{dt_1}\right) f = \frac{d}{dt_1}\left(f \circ (\gamma_1(t_1))\right) = \frac{d}{dt_1}\left(f \circ (\gamma(t(t_1)))\right)$$

$$= \frac{d}{dt}\left(f(\gamma(t))\right) \cdot \frac{dt}{dt_1} = d\gamma\left(\frac{d}{dt}\right) f \, \frac{dt}{dt_1}, \quad \forall f$$

$$\Rightarrow \dot{\gamma}_1(t_1) = \dot{\gamma}(t) \cdot \frac{dt}{dt_1} \qquad \square$$

Exercise: Let $\varphi: (M, g_M) \to (N, g_N)$ be an isometry

$\gamma \subset M$ $C^\infty$ curve $\Rightarrow \varphi(\gamma) \subset N$ $C^\infty$ curve

Prove: $\text{Length}_M(\gamma) = \text{Length}_N \varphi(\gamma)$

Arc length: parametrization.

$\gamma: [a, b] \to M$ $\qquad \frac{ds}{dt} = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$

$$t \in [a,b],\ s(t) = \text{Length}\left(\gamma|_{[a,t]}\right) = \int_a^t \sqrt{\langle \dot{\gamma}(\tau), \dot{\gamma}(\tau) \rangle}\, d\tau$$

$s: [a, b] \to [0, \text{Length}(\gamma|_{[a,b]})]$

$s = s(t), \quad t = t(s)$

Proposition: $\gamma = \gamma(t)$, $\gamma(s) = \gamma(t(s))$

$$\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\gamma(s)} \equiv 1$$

Proof:

$$\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\gamma(s)} = \left\langle \dot{\gamma}(t) \frac{dt}{ds}, \dot{\gamma}(t) \frac{dt}{ds} \right\rangle$$

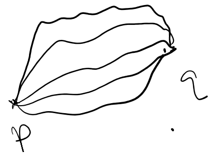
$$= \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \left(\frac{dt}{ds}\right)^2$$

$$= \left(\frac{ds}{dt}\right)^2 \left(\frac{dt}{ds}\right)^2 = \left(\frac{ds}{dt} \cdot \frac{dt}{ds}\right)^2$$

$$= 1 \qquad \square$$

$\text{Length}(\gamma)$, $\gamma$ piecewise $C^\infty$ curve

distance function: $(M^n, g)$



$$d: M \times M \to \mathbb{R}$$

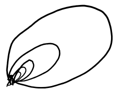
$$(p, q) \longmapsto d(p, q)$$

$$d(p, q) := \inf \{ \text{Length}(\gamma) \mid \gamma \in C_{p,q} \}$$

$C_{p,q} = \{\gamma : [a,b] \to M \mid \gamma$ piecewise smooth curves in $M$ with $\gamma(a) = p,\ \gamma(b) = q\}$

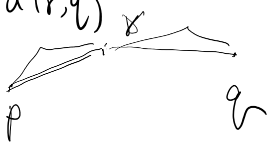
Check: (i) $d(p,q) \geq 0$, $\forall p,q \in M$

$d(p,q) = 0$



(ii) $d(p,q) = d(q,p)$

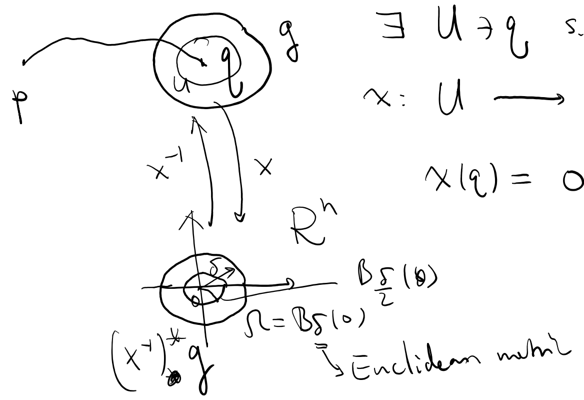
(iii) $d(p,q) \leq d(p,r) + d(r,q)$




? $d(p,q) = 0 \overset{?}{\Rightarrow} p = q$

Thm: $(M^n, d)$ is a metric space.

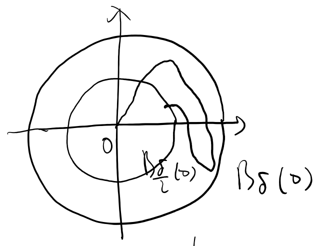
Proof: It remains to prove if $p \neq q$, then $\underline{\underline{d(p,q)>0}}$




$\exists\ U \ni q$ s.t. $p \notin U$

$x : U \longrightarrow \Omega = B_\delta(0)$

$x(q) = 0 \in \mathbb{R}^n$




Consider $\gamma : \underline{\underline{[a,b]}} \to B_\delta(0)$

s.t. $\gamma(a) = 0$, $\gamma(b) \in \partial B_{\frac{\delta}{2}}(0)$

piecewise smooth

Let $c \in [a,b]$ be the smallest number with

$\gamma(c) \in \partial B_{\frac{\delta}{2}}(0)$

$$d(p,q) \geq \text{Length}_{(x^{-1})^* g}(\gamma) \geq \text{Length}_{(x^{-1})^* g}(\gamma|_{[a,c]}) = \int_a^c \sqrt{\langle \dot\gamma, \dot\gamma \rangle_{(x^{-1})^* g}}\, dt \geq \int_a^c \sqrt{\varepsilon_1} \sqrt{\langle \dot\gamma, \dot\gamma \rangle_{g_{eucl.}}}\, dt$$

$$\boxed{\langle \dot\gamma, \dot\gamma \rangle_{(x^{-1})^* g} \geq \varepsilon \langle \dot\gamma, \dot\gamma \rangle_{g_{eucli.}}} \text{ on } B_\delta(0) \geq \sqrt{\varepsilon_1}\ \frac{\delta}{2} > 0$$

$$\| \qquad g_{ij} \dot\gamma^i \dot\gamma^j$$

$(g_{ij})$ $\qquad$ $(g_{ij}(x))$

$\lambda(x)$ smallest eig.

$\Lambda(x)$ greatest eig.

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continuous

$\Rightarrow$ Prop. $\underline{\varepsilon_1} \sum_i (\xi^i)^2 \le g_{ij}\xi^i\xi^j \le \underline{\varepsilon_2} \sum_i (\xi^i)^2$ on a cpt ~~set~~ domain. $\square$

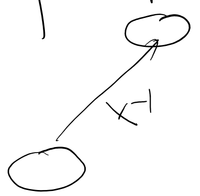
In local cpt coordinate neighborhood

$$\sqrt{\varepsilon_2}\,|x-y| \ge \hat{d}_g(x,y) \ge \sqrt{\varepsilon_1}\,|x-y|$$

Corollary: The topology of $M$ induced by distance function $d$, coincides the original topology of $M$.

Pf: $\square$

$f: M \to \mathbb{R}$

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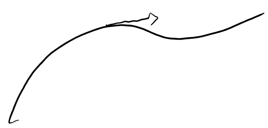
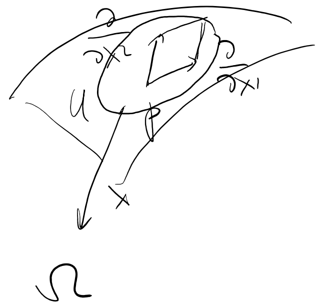
Riemannian Measure; Volume






$T_pM = \{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$

orthonormal basis

$\{e_1, \cdots, e_n\}$

$\mathrm{vol}(e_1, \cdots, e_n) = 1$

$\frac{\partial}{\partial x^i} = a_i^j e_j \qquad \det A > 0$

$$g_{ik} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \rangle = \langle a_i^j e_j, a_k^l e_l \rangle$$

$$= a_i^j a_k^l \delta_{jl} = \sum_{l=1}^n a_i^l a_k^l$$

$$= a_i^j a_k^l \delta_{jl} = \sum_{l=1} a_i^l a_k^l$$

Matrix: $\underbrace{(g_{ik}) = A A^T}$, $A = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & & \vdots \\ a_n^1 & a_n^2 & & a_n^n \end{pmatrix}$

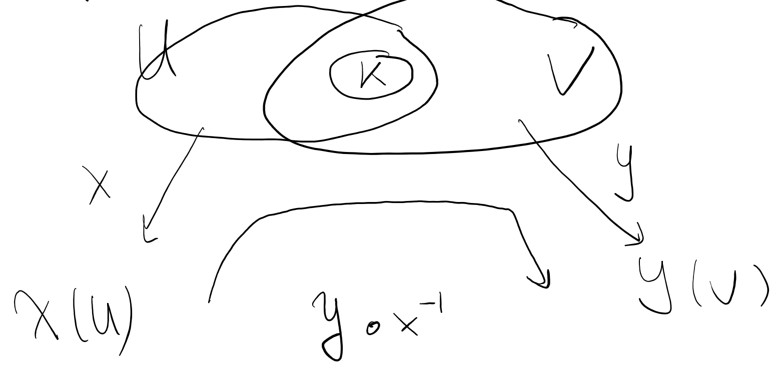
$$\mathrm{vol}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \det(a_i^j)\ \underline{\mathrm{vol}(e_1, \dots, e_n)}$$
$$= \det A$$
$$= \sqrt{\det(g_{ik})}$$

<u>Volume</u> of "smaller" compact domain.

$(U, x)$ chart, $\forall\ K \subset U$ cpt

$$\mathrm{vol}(K) = \int_{x(K)} \sqrt{\det(g_{ij})}\ \underbrace{dx^1 \cdots dx^n}_{\text{Lebesgue}} \qquad (*)$$

Well-defined?




$x^1, \dots, x^n \to y^1, \dots, y^n$

<u>Prop</u>: $(*)$ does not depend on the choices of charts.

<u>Pf</u>: $g_{ij}^x = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$, $g_{ij}^y = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle$

$$\frac{\partial}{\partial x^i} = \frac{\partial (y^k \circ x^{-1})}{\partial x^i} \cdot \frac{\partial}{\partial y^k}$$

<u>Exercise</u>: verify

Exercise: verify:

$$g^{x}_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \left\langle \frac{\partial (y^k \circ x^{-1})}{\partial x^i} \frac{\partial}{\partial y^k}, \frac{\partial (y^l \circ x^{-1})}{\partial x^j} \frac{\partial}{\partial y^l} \right\rangle$$

$$= \frac{\partial (y^k \circ x^{-1})}{\partial x^i} \frac{\partial (y^l \circ x^{-1})}{\partial x^j} g^{y}_{kl}$$

Matrix: $(g^{x}_{ij}) = (J)^T (g^{y}_{kl}) J$

$\square$

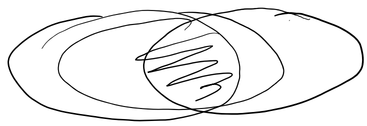
Volume of larger compact domain

$(M^n, g)$ locally finite covering $\{U_\alpha, x_\alpha\}$

partition of unity subordinate $\{U_\alpha\}_{\alpha \in A}$

$\phi_\alpha$

$\sum_\alpha \phi_\alpha = 1$



$K \subset M$

$$(*) \quad vol(K) := \sum_\alpha \int_{x_\alpha(K \cap U_\alpha)} \phi_\alpha \cdot \sqrt{\det(g^{x_\alpha}_{ij})}\, dx^1_\alpha \cdots dx^n_\alpha$$

Prop: (*) does not depend on the choice of the covering of the charts and partition of unity.

Proof: $\{V_\beta\}_{\beta \in B}$ $\quad y_\beta : V_\beta \to \Omega_\beta$

partition of unity $\{\psi_\beta\}_{\beta \in B}$

$$\sum_\beta \int_{y_\beta(K \cap V_\beta)} \psi_\beta \cdot \sqrt{\det(g^{y_\beta}_{ij})} \cdot dy^1_\beta \cdots dy^n_\beta$$

$$\sum_\beta \int_{y_\beta(K\cap V_\beta)} \psi_\beta \cdot \sqrt{\det(g_{ij}^{y_\beta})} \cdot dy_\beta^1 \cdots dy_\beta^n$$

$$= \sum_\beta \int_{\underline{y_\beta(K\cap V_\beta)}} \sum_\alpha \phi_\alpha \cdot \psi_\beta \cdot \sqrt{\det(g_{ij}^{y_\beta})}\, dy_\beta^1 \cdots dy_\beta^n$$

$$= \sum_\alpha \int_{\underline{x_\alpha(K\cap U_\alpha)}} \sum_\beta \psi_\beta \sqrt{\det(g_{ij}^{x_\alpha})}\, dx_\alpha^1 \cdots dx_\alpha^n$$

$$= \sum_\alpha \int_{x_\alpha(K\cap U_\alpha)} \sqrt{\det(g_{ij}^{x_\alpha})}\, dx_\alpha^1 \cdots dx_\alpha^n .$$

$\square$

下课

$$\sum_\beta \int_{y_\beta(K\cap V_\beta)} \psi_\beta \circ y_\beta^{-1} \sqrt{\det(g_{ij}^{y_\beta}) \circ y_\beta^{-1}}\, dy_\beta^1 \cdots dy_\beta^n$$

$$= \sum_\beta \int_{y_\beta(K\cap V_\beta)} \sum_\alpha \psi_\alpha \circ y_\beta^{-1}\, \psi_\beta \circ y_\beta^{-1} \sqrt{\det(g_{ij}^{y_\beta}) \circ y_\beta^{-1}}\, dy_\beta^1 \cdots dy_\beta^n$$

$$= \sum_\beta \sum_\alpha \int_{y_\beta(K\cap V_\beta \cap U_\alpha)} \psi_\alpha \circ y_\beta^{-1} \cdot \psi_\beta \circ y_\beta^{-1} \sqrt{\det(g_{ij}^{y_\beta}) \circ y_\beta^{-1}}\, dy_\beta^1 \cdots dy_\beta^n$$

finite sums

$$= \sum_\alpha \sum_\beta \int_{y_\beta(K\cap V_\beta \cap U_\alpha)} \psi_\alpha \circ y_\beta^{-1}\, \psi_\beta \circ y_\beta^{-1} \sqrt{\det(g_{ij}^{y_\beta}) \circ y_\beta^{-1}}\, dy_\beta^1 \cdots dy_\beta^n$$

change of variables

$$\sum_\alpha \sum_\beta \int_{x_\alpha(K\cap V_\beta \cap U_\alpha)} \psi_\alpha \circ x_\alpha^{-1} \cdot \psi_\beta \circ x_\alpha^{-1} \sqrt{\det(g_{ij}^{x_\alpha}) \circ x_\alpha^{-1}}\, dx_\alpha^1 \cdots dx_\alpha^n$$

$$= \sum_\alpha \int_{x_\alpha(K\cap U_\alpha)} \psi_\alpha \circ x_\alpha^{-1} \sqrt{\det(g_{ij}^{x_\alpha}) \circ x_\alpha^{-1}}\, dx_\alpha^1 \cdots dx_\alpha^n$$