Ricci curvature and Bochner formula

Ricci curvature:

① General Relativity: Einstein field equation

② Hamilton's Ricci flow

Perelman (Fields ~~medalist~~ medalist 2006)

③ Cédric Villani (Fields meldalist 2010)

optimal ~~transp~~ transportation.

Thm. Let (M,g) Rie mfld. $\forall f \in C^\infty(M)$, we have:

$$\boxed{\frac{1}{2}\Delta|\mathrm{grad} f|^2} = |\mathrm{Hess}\, f|^2 + \langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f)$$

Remark. $|\cdot|^2$, (U,x) $\quad \mathrm{grad} f = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}$

$$|\mathrm{grad} f|^2 = \langle \mathrm{grad} f, \mathrm{grad} f\rangle_g = g^{kl}\frac{\partial f}{\partial x^k}\frac{\partial f}{\partial x^l}.$$

$$|\mathrm{Hess} f|^2_{(p)} = g^{ik}g^{jl}\,\mathrm{Hess} f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\mathrm{Hess} f\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$= g^{ik}_{(p)} g^{jl}_{(p)} f_{;ij} f_{;kl}(p) \qquad \text{Hilbert - Schmidt norm}$$

Proof. Let $p \in M$. Pick a normal coordinate (U,x) of p.

$$\frac{1}{2}\Delta|\mathrm{grad} f|^2(p) = \frac{1}{2} g^{kl}_{(p)}\left(g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)_{;kl}(p)$$

$$= \frac{1}{2}\sum_{k=1}^n \left(g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)_{;kk}(p)$$

$$\overset{\nabla g \equiv 0}{=} \frac{1}{2}\sum_{k=1}^n \underbrace{g^{ij}(p)}_{\delta^{ij}}\left(\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)_{;kk}(p)$$

$$= \frac{1}{2}\sum_{k,i}\left[\left(\frac{\partial f}{\partial x^i}\right)^2\right]_{;kk}(p) = \frac{1}{2}\sum_{k,i}\left(2\frac{\partial f}{\partial x^i}\left(\frac{\partial f}{\partial x^i}\right)_{;k}\right)_{;k}(p)$$

$$= \sum_{k,i}\left[\left(\frac{\partial f}{\partial x^i}\right)_{;k}(p)\left(\frac{\partial f}{\partial x^i}\right)_{;k}(p) + \frac{\partial f}{\partial x^i}\left(\frac{\partial f}{\partial x^i}\right)_{;kk}(p)\right]$$

$$= \sum_{k,i}\left(f_{;ik}(p)\right)^2 + \sum_{k,i} f_{;i}(p)\, f_{;ikk}(p)$$

$$\sum\left(f_{;ik}(p)\right)^2 = |\mathrm{Hess} f|^2(p)$$

$$\sum_{k,i} \left(f_{;ik}(p)\right)^2 = |\mathrm{Hess} f|^2(p)$$

$T \equiv 0 \Rightarrow \quad f_{;ik} = f_{;ki} \qquad \mathrm{Hess} f$ symmetric (0,2)-tensor

$$= |\mathrm{Hess} f|^2(p) + \sum_{k,i} f_{;i}(p) f_{;ikik}(p)$$

Ricci Identity: $f_{;ikik} = f_{;kki} + f_{;h} R^{h}{}_{kik}$

$$= |\mathrm{Hess} f|^2(p) + \sum_{k,i} f_{;i}(p) f_{;kki}(p) + \sum_{k,i} f_{;i}(p) f_{;h}(p) R^{h}{}_{kik}(p)$$

$$\sum_{k,i} f_{;i}(p) f_{;kki}(p) = \sum_i f_{;i}(p) \Big(\sum_k f_{;kk}\Big)_{;i}(p) = \langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle(p), \qquad \langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle = g^{ij} f_{;i} (\Delta f)_{;j}$$

$$= |\mathrm{Hess} f|^2(p) + \langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle(p) + \underbrace{\sum_{k,i,h} f_{;i}(p) f_{;h}(p) R_{hkik}(p)}$$

$$\sum_{i,h} f_{;i} f_{;h} \Big(\sum_k R_{hkik}\Big)(p) \qquad \sum_{k,l} g^{kl} R_{hkil}(p) = \mathrm{Ric}_{hi}(p)$$

$$\mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f) = \mathrm{Ric}(g^{ij} f_{;i} \partial_j, g^{kl} f_{;k} \partial_l) = g^{ij} g^{kl} f_{;i} f_{;k} \mathrm{Ric}_{jl}$$

$$= |\mathrm{Hess} f|^2(p) + \langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle(p) + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f)(p).$$

Thm (Lichnerowicz 1958) (M^n, g) n-dim closed Rie. mfld. with Ricci $\geq (n-1)k > 0$ $\square$

Then $\lambda_1 \geq nk$.

$\lambda_1 - \lambda_0 \geq nk$, $\lambda_0 = 0$

Rmk: $\boxed{\mathrm{Ric} \geq (n-1) k\, g} \Leftrightarrow \mathrm{Ric}(X,X) \geq (n-1) k\, g(X,X), \ \forall X \in \Gamma(TM)$

Scaling behavior: $g \mapsto Ag$. $A > 0$

$$\mathrm{Ric}_{Ag}(X,X) = \mathrm{Ric}_{g}(X,X)$$

$$\mathrm{Ric}_{Ag}(X,X) = \mathrm{Ric}_g(X,X) \qquad \mathrm{Ric}_{Ag}(X,X) = \mathrm{Ric}_g(X,X) \geq \frac{(n-1)k}{A} Ag(X,X)$$

$$\underline{g^{ij} R_{kilj} X^k X^l} = \underline{g^{ij}}\, g_{km} R^m{}_{ilj}\, \underline{X^k X^l}.$$

$$\Delta_g f = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i}\left(\sqrt{G}\, g^{ij} \frac{\partial f}{\partial x^j}\right)$$

$$\Delta_{Ag} = \frac{1}{A}\Delta_g \Rightarrow \lambda_1(Ag) = \frac{1}{A}\lambda_1(g). \qquad \square$$

Proof: Integrating the Bochner formula over M, $\Delta f + \lambda_1 f = 0$

$$0 = \int_M \frac{1}{2}\Delta |\mathrm{grad} f|^2 d\mu = \int_M |\mathrm{Hess} f|^2 d\mu + \int_M \langle \mathrm{grad}\, \Delta f, \mathrm{grad} f\rangle d\mu + \int_M \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f) d\mu$$

$$\Rightarrow \boxed{\int_M |\mathrm{Hess} f|^2 d\mu} - \lambda_1 \int_M |\mathrm{grad} f|^2 d\mu + (n-1)k \int_M |\mathrm{grad} f|^2 d\mu$$

$$\geq ((n-1)k - \lambda_1) \int_M |\mathrm{grad} f|^2 d\mu + \frac{1}{n}\int_M (\Delta f)^2 d\mu$$

$$= \left((n-1)k - \lambda_1 + \frac{\lambda_1}{n}\right) \int_M |\mathrm{grad} f|^2 d\mu \qquad |\mathrm{grad} f|^2 \neq 0$$

$$f \equiv \mathrm{const} \Rightarrow \langle \mathrm{grad} f, X\rangle = Xf = 0 \Rightarrow \mathrm{grad} f = 0$$

$$\Rightarrow \lambda_1 \geq (n-1)k$$

Claim: $|\mathrm{Hess} f|^2(p) \geq \frac{1}{n}(\Delta f)^2(p), \quad \forall p \in M.$

normal cood. around p.

$$|\mathrm{Hess} f|^2(p) = \sum_{i,k} \underline{(f_{;ik})^2(p)} \geq \underline{\sum_{i=1}^n (f_{;ii}(p))^2}$$

$$\underset{\text{Cauchy-Schwarz}}{\geq} \frac{1}{n}\left(\sum_{i=1}^n f_{;ii}(p)\right)^2 = \frac{1}{n}(\Delta f(p))^2 \qquad \square$$

$$(\Delta f)^2 = (\lambda_1 f)^2 = \lambda_1 f \cdot (\lambda_1 f) = \lambda_1 f \cdot (-\Delta f)$$

$$\frac{1}{n}\int_M (\Delta f)^2 d\mu = -\frac{\lambda_1}{n} \underline{\int_M f \Delta f\, d\mu} = \frac{\lambda_1}{n}\int_M |\mathrm{grad} f|^2 d\mu$$

$$0 \geq \left((n-1)k - \lambda_1 + \frac{\lambda_1}{n} \right) = (n-1)k - \frac{n-1}{n}\lambda_1$$

$$\Longrightarrow \quad \lambda_1 \geq nk. \qquad \square$$

$$Ric \geq (n-1)k \xrightarrow{\text{Bochner}}$$

休息 14:15

$$\boxed{\frac{1}{2}\Delta\left(|\operatorname{grad} f|^2\right) - \langle \operatorname{grad} \Delta f, \operatorname{grad} f\rangle \geq \frac{1}{n}(\Delta f)^2 + (n-1)k\,|\operatorname{grad} f|^2}$$

$$\forall f \in C^\infty(M)$$

Thm. (M^n, g). Let $p \in M$.

$$\boxed{\frac{1}{2}\Delta|\operatorname{grad} f|^2_{(p)} - \langle \operatorname{grad}\Delta f, \operatorname{grad} f\rangle(p) \geq \frac{1}{n}(\Delta f)^2(p) + K(p)|\operatorname{grad} f|^2(p), \ \forall f \in C^\infty(M)}$$

$$\Longleftrightarrow \quad Ric(p) \geq K(p)\, g(p), \ \forall p \in M. \qquad Ric(Z,Z)(p) \geq K(p)\, g(Z,Z)(p)$$

$$\forall Z \in \Gamma(TM)$$

Proof. "$\Longleftarrow$" Bochner.

"$\Longrightarrow$" Bochner formula. $\forall f \in C^\infty(M)$

$$|\operatorname{Hess} f|^2(p) + Ric(\operatorname{grad} f, \underbrace{\operatorname{grad} f}_{Z})(p) \geq \frac{1}{n}(\Delta f)^2(p) + K(p)|\operatorname{grad} f|^2(p)$$

(U, x) normal coord. around p.

$$(1) \quad \sum_{i,j} f_{;ij}(p)^2 + \sum_{i,j} Ric_{ij}(p)\, f_{;i}(p)\, f_{;j}(p) \geq \frac{1}{n}\left(\sum_i f_{;ii}(p)\right)^2 + K(p)\sum_i f_{;i}^2(p)$$

$$\forall f \in C^\infty(M)$$

Observe $\forall Z \in \Gamma(TM)$, $Z_p = Z^i \frac{\partial}{\partial x^i} \in T_pM$

$\forall$ $n \times n$ symmetric matrix $Y = (Y_{ij})$

We can always construct a function $f \in C^\infty(M)$, s.t.

$$\begin{cases} f_{;i}(p) = Z^i, & i = 1, \dots, n \\ f_{;ij}(p) = Y_{ij}, & i,j = 1, \dots, n \end{cases} \qquad Y_{ij} \equiv 0$$

$(1) \Rightarrow \forall Z_p \in T_pM$, $\forall Y = (Y_{ij})$ $n \times n$ sym. matrix

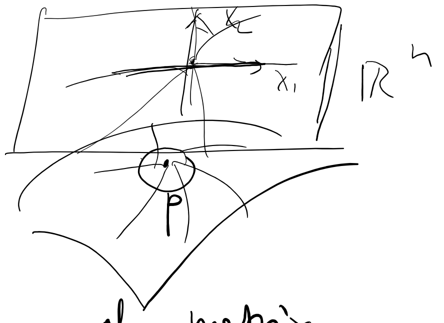
$$\|Y\|^2 + \sum_{i,j} Ric_{ij} Z^i Z^j \geq \frac{1}{n}(\operatorname{tr} Y)^2 + K(p)\sum_i (Z^i)^2.$$

$$(1) \Longleftrightarrow \|Y\|^2 + \sum (Ric_{ij} - K(p)\delta_{ij}) Z^i Z^j \geq \frac{1}{n}(\operatorname{tr} Y)^2$$

$\|Y\|^2 + \sum_{i,j} \mathrm{Ric}_{ij} z^i z^j \geq \frac{1}{n}(\mathrm{tr}\,Y)^2 + K(p) \ldots$

(2) $\iff \|Y\|^2 + \sum_{i,j}\left(\mathrm{Ric}_{ij} - K(p)\,\underbrace{\delta_{ij}}_{g_{ij}(p)}\right) z^i z^j \geq \frac{1}{n}(\mathrm{tr}\,Y)^2.$

where $\|Y\|^2 = \sum_{i,j} Y_{ij}^2$



normal coord.

$\exists$ a normal coord. (U, x') s.t. (Y'_{ij}) is a diagonal matrix.

Denote $\Lambda = (\Lambda_i) = (Y'_{ii})$

$$\boxed{\begin{aligned}&\|\Lambda\|^2 + \sum_{i,j}\left(\mathrm{Ric}'_{ij} - g'_{ij}(p)K(p)\right) z'^i z'^j \geq \frac{1}{n}\Big(\sum_i \Lambda_i\Big)^2 \\ &\qquad \forall\, \Lambda \in \mathbb{R}^n,\ \forall\, z_p \in T_pM.\end{aligned}}$$

(2) $\Rightarrow$ (3)

$\|\Lambda\|^2 \geq \frac{1}{n}\left(\sum_i \Lambda_i\right)^2$, "=" $\iff \Lambda_i = \Lambda_j,\ \forall i, j$.

Posing $\sum_i \Lambda_i = t$ $\quad \|\Lambda\|^2 \geq \frac{1}{n}t^2$

(3) $\xrightarrow{\text{Set } \Lambda_i = \frac{1}{n}t}$ $\sum_{i,j}\left(\mathrm{Ric}'_{ij} - g'_{ij}(p)K(p)\right) z'^i z'^j \geq 0.$

$\Updownarrow$

$\mathrm{Ric}(Z, Z)(p) \geq K(p)\, g(Z, Z)(p).$ $\square$

Rmk. Exercise.

$\frac{1}{2}\Delta|\mathrm{grad} f|^2 - \langle \mathrm{grad} f, \mathrm{grad}\,\Delta f\rangle \geq K\,|\mathrm{grad} f|^2$ at p. $\forall f \in C^\infty(M)$

$\iff \mathrm{Ric}(p) \geq K(p)$.

Bakry – Émery: Γ - calculus.

Γ, Γ_2 $\quad \Gamma_0(f, g) = fg$.

$\forall f, g \in C^\infty(M)$, $\Gamma(f, g) := \frac{1}{2}\big(\Delta(fg) - f\Delta g - g\Delta f\big).$

i.e. $\Gamma(f,f) = \underbrace{|\text{grad} f|^2}$

$= \langle \text{grad} f, \text{grad}\, g\rangle$

$$\Gamma_2(f,g) = \frac{1}{2}\left(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\right)$$

iteratively

$$\Gamma_2(f,f) = \frac{1}{2}\Delta\Gamma(f,f) - \Gamma(f,\Delta f).$$

$$= \frac{1}{2}\Delta|\text{grad} f|^2 - \langle \text{grad} f, \text{grad}\,\Delta f\rangle.$$

(M^n, g, Δ)

$$\boxed{\Gamma_2(f,f) \geq \frac{1}{n}(\Delta f)^2 + K\Gamma(f,f), \ \forall f \in C^\infty(M)}$$

$\Longleftrightarrow \ Ric \geq K.$

Thm: (M^n, g) closed with $Ricci \geq (n-1)k > 0$

~~$\Rightarrow \lambda_1 \geq nk$~~ If $\lambda_1 = nk$

then we have $\text{diam}(M^n, g) \geq \frac{\pi}{\sqrt{k}}$.

Proof: $g \mapsto Ag$ $\qquad Ric_{Ag} \geq \frac{(n-1)k}{A} \cdot Ag$

$\lambda_1(Ag) = \frac{1}{A}\lambda_1(g)$

$\text{diam}(M^n, Ag) = \sqrt{A}\,\text{diam}(M^n, g)$

Set $A = k$.

$$\boxed{\begin{array}{l} Ric_{Ag} \geq (n-1), \quad \lambda_1(Ag) = n, \\ \Rightarrow \text{diam}(M^n, Ag) \geq \pi \end{array}}$$

$$\Delta f + \overset{\lambda}{\overset{\shortparallel}{n}} f = 0$$

$$\frac{1}{2}\Delta|\text{grad} f|^2 \geq \frac{1}{n}(\Delta f)^2 + \langle \text{grad}\,\Delta f, \text{grad} f\rangle$$
$$+ (n-1)|\text{grad} f|^2$$

$$= nf^2 - n|\text{grad} f|^2 + (n-1)|\text{grad} f|^2$$

$$= \underset{f(nf)}{\underline{nf^2}} - |\text{grad} f|^2 = -f\Delta f - |\text{grad} f|^2$$

$$= -\frac{1}{2}\Delta(f^2)$$

$$= \underbrace{nf}_{f(nf)} - |\mathrm{grad} f|^2 \qquad - f\Delta f - |\mathrm{grad} f|^2 = -\frac{1}{2}\Delta(f^2)$$

$$\left.\begin{aligned} \Rightarrow \quad & \frac{1}{2}\Delta\left(|\mathrm{grad} f|^2 + f^2\right) \geq 0 \\ & \frac{1}{2}\int \Delta\left(|\mathrm{grad} f|^2 + f^2\right) = 0 \end{aligned}\right\} \Rightarrow \Delta\left(|\mathrm{grad} f|^2 + f^2\right) = 0$$

$$\Rightarrow \boxed{|\mathrm{grad} f|^2 + f^2 \equiv \text{const.}}$$

normalize f s.t. $\max\limits_{p \in M} (f(p))^2 = 1$.

Observation. at the maximal / minimum pt of f.

we have $\mathrm{grad} f = 0$
$\Uparrow$
$$(\langle \mathrm{grad} f, X \rangle = Xf = 0)$$

$$\Rightarrow |\mathrm{grad} f|^2 + f^2 \equiv 1 \Rightarrow \max_{p \in M} f(p) = 1,$$

$$\min_{p \in M} f(p) = -1$$

$\exists\ p, q$ s.t. $f(p) = 1$, $f(q) = -1$

Let γ be a normal minimizing geodesic from p to q.

$$\left|\frac{d}{dt} f \circ \gamma(t)\right| = |\langle \mathrm{grad} f, \dot\gamma(t) \rangle|$$
$$\leq |\mathrm{grad} f(\gamma(t))|$$
$$= \sqrt{1 - f(\gamma(t))^2}$$

$$\underbrace{\int_0^{d(p,q)} \frac{\left|\frac{d}{dt} f(\gamma(t))\right|}{\sqrt{1 - f(\gamma(t))^2}}}_{\geq} \leq \int_0^{d(p,q)} 1 = d(p,q)$$

$$\geq \left| \int_0^{d(p,q)} \frac{\frac{d}{dt} f(\gamma(t))}{\sqrt{1 - f(\gamma(t))^2}}\, dt \right|$$

$$= \left| \int^1 \frac{dx}{\quad} \right| = $$

$$= \left| \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \right| = \pi \quad \square$$

下课.