第二十一讲

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The Second Variation : Revisited.

Let γ be a C^∞ curve. $\gamma : [a,b] \to M$.

Variational field $V(t) = \frac{\partial F}{\partial v}(t, 0)$

Velocity field $T(t) := \dot\gamma(t)$

F: a variation of γ $F : [a,b]\times(-\varepsilon,\varepsilon) \to M$

$(t, v) \longmapsto F(t,v)$

(F C^∞, $F(t,0) = \gamma(t)$, $\forall t \in [a,b]$)

Energy functional $\rightsquigarrow$ $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$

$v \longmapsto E(v)$

$$E(v) := \frac{1}{2}\int_a^b \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle(t,v)\, dt$$

1^{st} and 2^{nd} variation formula.

(1) $$E'(0) = \langle V, T\rangle\Big|_a^b - \int_a^b \langle V, \boxed{\nabla_T T}\rangle dt$$

(2) $$E''(0) = \langle \nabla_V V, T\rangle\Big|_a^b - \int_a^b \langle \nabla_V V, \nabla_T T\rangle dt + \int_a^b \Big(\langle \nabla_T V, \nabla_T V\rangle + \underbrace{\langle R(V,T)V, T\rangle}_{-\langle R(V,T)T, V\rangle}\Big) dt$$

Remark: $\nabla g \equiv 0$, torsion free

$$\frac{d}{dv}E(v) = \frac{1}{2}\int_a^b \frac{d}{dv}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle dt \overset{\nabla g \equiv 0}{=} \int_a^b \left\langle \frac{D}{\partial v}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle dt$$

$$\overset{\text{torsion free}}{=} \int_a^b \left\langle \frac{D}{\partial t}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle dt \quad \boxed{(A)}$$

$$\overset{\nabla g \equiv 0}{=} \int_a^b \frac{d}{dt}\left\langle \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle - \left\langle \frac{\partial F}{\partial v}, \frac{D}{\partial t}\frac{\partial F}{\partial t}\right\rangle dt$$

$$\Rightarrow E'(0) = \langle V, T\rangle\Big|_a^b - \int_a^b \langle V, \nabla_T T\rangle dt$$

$$\frac{d^2}{dv^2}E(v)\Big|_{v=0} = \int_a^b \frac{d}{dv}\left\langle \frac{D}{\partial t}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle dt\,\Big|_{v=0}$$

$\frac{D}{\partial t}\frac{\partial F}{\partial v}$ $\shortparallel$

$$\overset{\nabla g \equiv 0}{=} \int_a^b \left\langle \frac{D}{\partial v}\frac{D}{\partial t}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle + \left\langle \frac{D}{\partial t}\frac{\partial F}{\partial v}, \frac{D}{\partial v}\frac{\partial F}{\partial t}\right\rangle dt\Big|_{v=0}$$

$$\underset{\text{Ricci Identity}}{\overset{\text{torsion free}}{=}} \int_a^b \langle \nabla_T V, \nabla_T V\rangle dt + \int_a^b \left\langle \frac{D}{\partial t}\frac{D}{\partial v}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right\rangle$$

$$\underset{\text{Ricci Identity}}{=} \underline{\int_a^b \langle \nabla_T V, \nabla_T V\rangle dt} + \int_a^b \langle \frac{D}{\partial t}\frac{D}{\partial v}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle + \langle R\left(\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\right)\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle \Big|_{v=0} dt$$

$$\overset{\nabla g\equiv 0}{=} \int_a^b \langle \nabla_T V, \nabla_T V\rangle dt + \underbrace{\int_a^b \frac{d}{dt}\langle \frac{D}{\partial v}\frac{\partial F}{\partial v}, \frac{\partial F}{\partial t}\rangle}\ - \langle \frac{D}{\partial v}\frac{\partial F}{\partial v}, \frac{D}{\partial t}\frac{\partial F}{\partial t}\rangle dt \Big|_{v=0}$$

$$+ \int_a^b \langle R(V,T)V, T\rangle dt$$

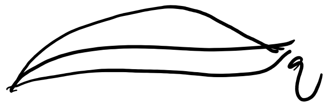
$$E''(0) = \int_a^b \left(\langle \nabla_T V, \nabla_T V\rangle - \underline{\langle R(V,T)T, V\rangle}\right) dt$$

$$\frac{D}{\partial v}\frac{\partial F}{\partial v}\Big|_{\substack{t=a,b\\ v=0}} + \underbrace{\langle \nabla_V V, T\rangle\Big|_a^b}_{=0} - \int_a^b \langle \nabla_V V, \underbrace{\boxed{\nabla_T T}}_{=0}\rangle dt \qquad \square$$

$\underline{E'(0)=0}$, & $E''(0) \geq 0$ $\rightarrow$ 'local' minimum

$\underline{(a)}$ $F(a,v) = F(a,0)$, $F(b,v) = F(b,0)$, $\forall v \in (-\varepsilon, \varepsilon)$

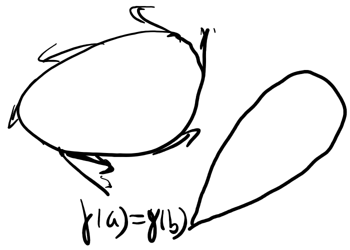
$E'(0)=0 \Rightarrow \nabla_T T \equiv 0$, $\Rightarrow \gamma$ is a geodesic



$$E''(0) = \int_a^b \left(\underbrace{\langle \nabla_T V, \nabla_T V\rangle}_{\geq 0} \underbrace{- \langle R(V,T)T, V\rangle}_{\geq 0}\right) dt \ \cancel{\geq}\ 0$$

Sec ≤ 0

(b) $\gamma : [a,b] \to M$ C^∞ closed curve



$\gamma(a) = \gamma(b)$, $\dot\gamma(a) = \dot\gamma(b)$

$E'(0)=0 \Rightarrow \nabla_T T = 0$. γ is a closed geodesic

$$E''(0) = \int_a^b \left(\underbrace{\langle \nabla_T V, \nabla_T V\rangle}_{\geq 0} - \underline{\langle R(V,T)T, V\rangle}\right) dt \ \cancel{\geq}\ 0$$

Sec ≤ 0

(a),(b), $\underline{\text{Sec} \leq 0}$, $\Rightarrow$ $\underline{E'(0)=0}$, $\underline{E''(0) \geq 0}$

$\Updownarrow$

γ is geodesic

'local' minimum

Sec $= 0$

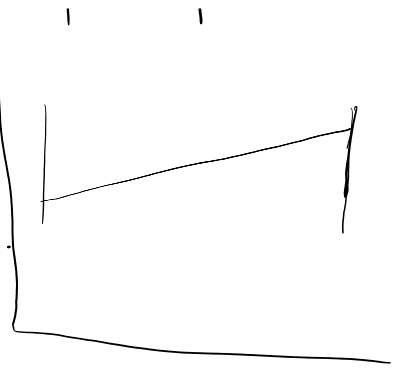
$E'(0) = 0$

$E''(0) \geq 0$

· 'local' minimum.

· Length $L(v): (-\varepsilon, \varepsilon) \to \mathbb{R}$

$v \longmapsto L(v)$



γ is parametrized proportionally to arc length.

If $\underline{L(0) \leq L(v)}$, $\forall v \in (-\varepsilon, \varepsilon)$

$$\Rightarrow 2(b-a)E(0) = L(0) \leq \underset{\sim}{L(v)} \leq 2(b-a)E(v) \Rightarrow E(0) \leq E(v).$$

$\underline{(a), (b)}$ $\boxed{E'(0) = 0, \quad E''(0) < 0} \Rightarrow$ energy of γ is $\underline{\text{not}}$ (locally) minimal.

$\Updownarrow$

γ is geodesic

'global'

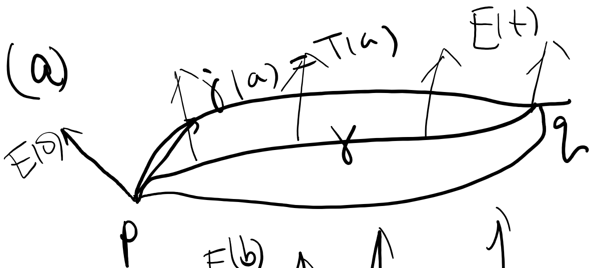
$\Rightarrow$ the geodesic γ is $\underline{\text{not}}$ minimizing.

$$E''(0) = \int_a^b (\underbrace{\langle \nabla_T V, \nabla_T V \rangle}_{\geq 0} - \underbrace{\langle R(V,T)T, V \rangle}_{\text{sec} > 0}) \, dt \underline{\,<\, 0}$$

$\boxed{\text{sec} > 0}$ $\quad \underbrace{\int_a^b \langle \nabla_T V, \nabla_T V \rangle \, dt}_{\overset{!}{0}} \;\overset{?}{<}\; \underbrace{\int_a^b \langle R(V,T)T, V \rangle \, dt}_{>0}$

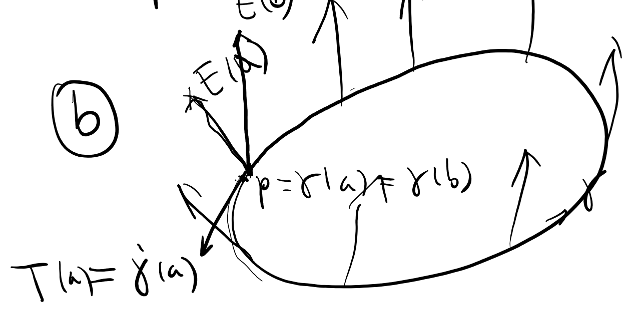
$\underline{\text{Particular case}}$: $\boxed{V \text{ is parallel along } \gamma, \text{ i.e. } \nabla_T V \equiv 0}$

(a)



$$\frac{V(p) = V(q) = 0}{f(t)E(t)}$$

(b)



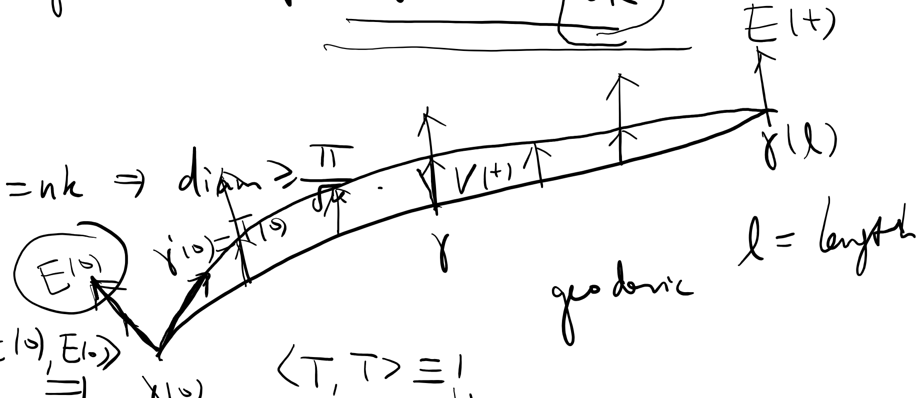
$E(b) \overset{?}{\neq} E(a)$

T

$\underline{\text{休息到 } 14:50}$.

Lemma. (Bonnet 1855 (surfaces) and Synge 1926) (M, g) Rie. mfld.

Lemma (Bonnet 1855 (surfaces) and Synge 1926) (M,g) Rie. mfld. $\sec \geq k > 0$. Then geodesics of length $> \frac{\pi}{\sqrt{k}}$ can not be minimizing.

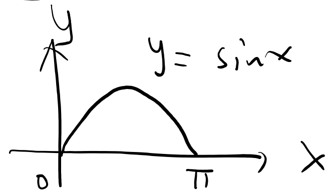


Recall: $Ric \geq (n-1)k$, $\lambda_1 = nk \Rightarrow diam \geq \frac{\pi}{\sqrt{k}}$.

Proof: $E(0) \in T_{\gamma(0)}M$, $\langle E(0), E(0)\rangle = 1$

s.t. $\langle E(0), T(0)\rangle = 0$ $\qquad \langle T, T\rangle \equiv 1$

$$E(t) := P_{\gamma;0,t}(E(0))$$



$$V(t) := \sin\left(\frac{\pi}{l}t\right) E(t) \qquad V(0) = V(l) = 0$$

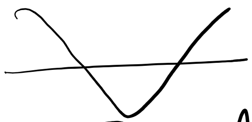
$$E''(0) = \int_0^l \left(\langle \nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle\right) dt$$

$$\int_0^l \langle \nabla_T V, \nabla_T V\rangle dt = \int_0^l \left\langle \frac{\pi}{l}\cos\left(\frac{\pi}{l}t\right)E(t), \frac{\pi}{l}\cos\left(\frac{\pi}{l}t\right)E(t)\right\rangle dt$$

$$= \boxed{\left(\frac{\pi}{l}\right)^2 \int_0^l \cos^2\left(\frac{\pi}{l}t\right) dt} \qquad \square$$

$$\int_0^l \langle R(V,T)T, V\rangle dt = \int_0^l \sin^2\left(\frac{\pi}{l}t\right) \underbrace{\langle R(E,T)T, E\rangle}_{K(E,T)} dt$$

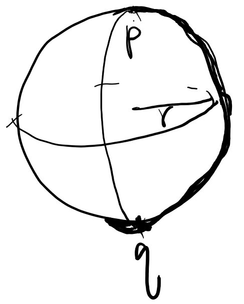
$$\geq \boxed{k \int_0^l \sin^2\left(\frac{\pi}{l}t\right) dt}$$



$$\boxed{\int_0^l \sin^2\left(\frac{\pi}{l}t\right) dt = \int_0^l \cos^2\left(\frac{\pi}{l}t\right) dt} \qquad \underbrace{\int_0^l \cos\left(\frac{2\pi}{l}t\right) dt}_{=0}$$

Hope. $k > \cancel{0} \left(\frac{\pi}{l}\right)^2 \Longleftrightarrow l > \frac{\pi}{\sqrt{k}}$.

$$l > \frac{\pi}{\sqrt{k}} \Rightarrow E''(0) < 0 \Rightarrow \gamma \text{ is not minimizing} \quad \square$$



$S^2 \quad k \quad \boxed{r = \frac{1}{\sqrt{k}}}$

$l > \pi \cdot \frac{1}{\sqrt{k}}$

<u>Remark on the Proof</u>: the choice of $E(0)$ is not unique.

$$T_{\gamma(0)}M : \{\gamma'(0) = T(0), \underbrace{E_2(0), \ldots, E_n(0)}_{E(0)}\} \text{ orthonormal}$$

<u>Lemma'</u> (Myers 1941) $\underline{(M, g)}$ $\boxed{Ric \geq (n-1)k > 0}$

Then ~~the~~ geodesics of <u>length $> \frac{\pi}{\sqrt{k}}$</u> ~~…~~ can not minimizing.

<u>Proof</u>. $E_i(0) \xrightarrow{\text{Parallel}} E_i(t)$

$$\underline{V_i(t)} = \sin\left(\frac{\pi}{l}t\right) E_i(t), \quad i = 2, \ldots, n.$$

$$\frac{\partial^2}{\partial v_i^2} E(v_i)\Big|_{v_i=0} < \underline{k \int_0^l \cos^2\left(\frac{\pi}{l}t\right) dt} \qquad i = 2, \ldots, n$$

$$l > \frac{\pi}{\sqrt{k}} \Rightarrow \left(\frac{\pi}{l}\right)^2 < k \qquad - \int_0^l \sin^2\left(\frac{\pi}{l}t\right) K(E_i, T) dt.$$

$$i = 2, \ldots, n$$

$$\underbrace{\sum_{i=2}^n \frac{\partial^2}{\partial v_i^2} E(v_i)\Big|_{v_i=0}} < (n-1)k \int_0^l \cos^2\left(\frac{\pi}{l}t\right) dt - \int_0^l \sin^2\left(\frac{\pi}{l}t\right) \underbrace{\sum_{i=2}^n K(E_i, T)}_{Ric(T)} dt$$

$$\leq (n-1)k \underbrace{\left(\int_0^l \cos^2\left(\frac{\pi}{l}t\right) dt - \int_0^l \sin^2\left(\frac{\pi}{l}t\right) dt\right)}$$

$$= 0$$

$$\Rightarrow \exists\, i_0 \text{ s.t. } \frac{\partial^2}{\partial v_{i_0}^2} E(v_{i_0})\Big|_{v_{i_0}=0} < 0.$$

$\Rightarrow \exists\, i_0$ s.t. $\frac{\partial}{\partial v_{i_0}^2} E(v_{i_0}) \Big|_{v_{i_0}=0} < 0.$

$\Longrightarrow \gamma$ is not minimizing. $\square$

Hopf – Rinow. completeness

p q

sec $\geq k > 0$ Ric $\geq (n-1)k > 0$

Bonnet – Myers Theorem.

Cor. (Myers 1935 / 1941 Duke Math. J.)

(M, g) ① complete Rie. mfld.

② $\mathrm{Ric} \geq (n-1)k > 0$

Then $\mathrm{diam}(M, g) \leq \frac{\pi}{\sqrt{k}}$.

Proof. If not

p q

$d(p,q) > \frac{\pi}{\sqrt{k}}$

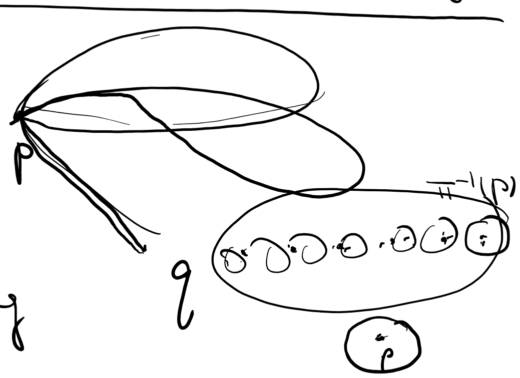
$\Downarrow$

M bounded closed $\Rightarrow$ M compact.

Furthermore, M has finite fundamental group.

$M,\ \pi_1(M)$

$\pi_1(M, p)$



$M \quad \tilde{M}$ universal covering

$\pi : \tilde{M} \to M$

$\pi_1(M, p)$ 1-1 correspondence to $\pi^{-1}(p)$

Proof: $(\tilde{M}, \tilde{g})$ complete Rie. $\widetilde{\mathrm{Ric}} \geq (n-1)k \Rightarrow \tilde{M}$ compact

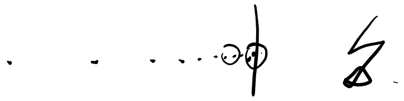
$\downarrow \pi$ covering space

local isometry

(M, g) complete, $\mathrm{Ric} \geq (n-1)$ ---

If $\#\pi^{-1}(p) = \infty \to$ limit point.

If $\#\pi^{-1}(p) = \infty \Rightarrow$ limit point.



Then $\#\pi^{-1}(p) < \infty$. $\square$

compact $\Rightarrow |\pi_1(M)| < \infty$

- $Ric \geq (n-1)k > 0 \Rightarrow \lambda_1 \geq nk$

- compact, $Ric \geq (n-1)k > 0$, $\lambda_1 = nk$

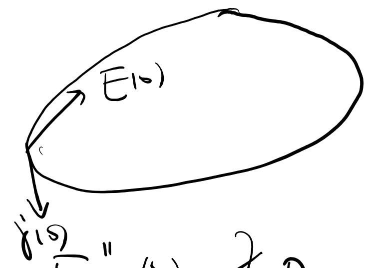
$\Rightarrow \operatorname{diam}(M, g) \geq \frac{\pi}{\sqrt{k}}$

Cor. (M, g) compact Rie. mfld. $Ric \geq (n-1)k > 0$

$\lambda_1 = nk$ Then $\operatorname{diam}(M, g) = \frac{\pi}{\sqrt{k}}$.

Q: When does the "=" holds in the Bonnet Myers diameter estimates? 郑绍远

(b) $sec > 0$



γ a closed geodesic

$\gamma : [0, l] \to M$.

$\gamma(0) = \gamma(l)$, $\gamma'(0) = \gamma'(l)$

$E''(0) \not< 0$

$$\boxed{\exists\, E(0) \in T_{\gamma(0)}M, \text{ s.t. } \langle E(0), \gamma'(0) \rangle = 0, \ \langle E(0), E(0) \rangle = 1 \text{ and } P_{\gamma,0,l}(E(0)) \neq E(0)}$$

$P_{\gamma;0,l} : T_{\gamma(0)}M \longrightarrow T_{\gamma(l)}M = T_{\gamma(0)}M$

linear transformation

eigenvalue λ P_γ

linear transformation

$\underline{+1}$ is an eigenvalue of $\underline{P_{\gamma;0,l}}$.

$$P_{\gamma,0,l}(\dot\gamma(0)) = \dot\gamma(l) = \dot\gamma(0)$$

$$\boxed{\text{\underline{Problem} Does the eigenvalue } +1 \text{ of } P_{\gamma;0,l} : T_{\gamma(0)}M \to T_{\gamma(0)}M \text{ have multiplicity } \geq 2 \text{ ?}}$$

Synge Theorem

下课